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ON THE MOREAU–YOSIDA REGULARIZATION OF THE VECTOR k-NORM RELATED FUNCTIONS*

BIN WU†, CHAO DING‡, DEFENG SUN§, AND KIM-CHUAN TOH†

Abstract. Matrix optimization problems (MOPs) involving the Ky Fan k-norm arise frequently from many applications. In order to design algorithms to solve large scale MOPs involving the Ky Fan k-norm, we need to understand the first and second order properties of the Moreau–Yosida regularization of the Ky Fan k-norm function and the indicator function of the Ky Fan k-norm ball. According to the general theory on spectral functions, in this paper we shall conduct a thorough study on the Moreau–Yosida regularization of the vector k-norm function and the indicator function of the vector k-norm ball. In particular, we show that the proximal mappings associated with these two vector k-norm related functions both admit fast and analytically computable solutions. Moreover, we propose algorithms of low computational cost to compute the directional derivatives of these two proximal mappings and then completely characterize their Fréchet differentiability. The work here thus builds the fundamental tools needed in the design of proximal point based algorithms for solving large scale MOPs involving the Ky Fan k-norm as well as in the study of the sensitivity and stability analysis of these problems.

Key words. Moreau–Yosida regularization, Ky Fan k-norm, metric projector

AMS subject classifications. 90C25, 90C30, 65K05, 49J52

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1. Introduction.

Let $f : \mathcal{Z} \to (-\infty, +\infty]$ be a closed proper convex function defined on a finite dimensional real Euclidean space $\mathcal{Z}$ equipped with an inner product $\langle \cdot, \cdot \rangle_{\mathcal{Z}}$ and its induced norm $\| \cdot \|_{\mathcal{Z}}$. The Moreau–Yosida regularization of f at $x \in \mathcal{Z}$ is defined by

$$\chi_f(x) := \min_{z \in \mathcal{Z}} \left\{ f(z) + \frac{1}{2} \|z - x\|_{\mathcal{Z}}^2 \right\}. \tag{1}$$

The unique optimal solution $P_f(x)$ to (1) is called the proximal point of x associated with f, and $P_f : \mathcal{Z} \to \mathcal{Z}$ is called the proximal mapping. It is known that $P_f(\cdot)$ is globally Lipschitz continuous with modulus 1 [36, Proposition 12.19] and $\chi_f(\cdot)$ is continuously differentiable on $\mathcal{Z}$ [26] (see also [35, Theorem 31.5]) with

$$\nabla \chi_f(x) = x - P_f(x), \quad x \in \mathcal{Z}. \tag{2}$$

Denote the Fenchel conjugate of f by $f^*(x) := \sup_{z \in \mathcal{Z}} \left\{ \langle x, z \rangle_{\mathcal{Z}} - f(z) \right\}$ for any $x \in \mathcal{Z}$. A particularly useful property for the Moreau–Yosida regularization is the following

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†Department of Mathematics, National University of Singapore, Singapore 119076 (wubin@nus.edu.sg, mattohkc@nus.edu.sg).
‡National Center for Mathematics and Interdisciplinary Sciences, Chinese Academy of Sciences, P. R. China (dingchao@amss.ac.cn). Part of this work was done while the author was with the Department of Mathematics, National University of Singapore. The research of this author was supported by the National Science Foundation for Distinguished Young Scholars of China (grant 11301515).
§Department of Mathematics and Risk Management Institute, National University of Singapore, Singapore 119076 (matsundf@nus.edu.sg). The research of this author was supported in part by the Academic Research Fund under grant R-146-000-149-112.