Characterization of the Robust Isolated Calmness for a Class of Conic Programming Problems

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This is a joint work with Defeng Sun at National University of Singapore and Liwei Zhang at Dalian University of Technology.

Optimization problem

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\begin{align*}
\min \quad & f(x) \\
\text{s.t.} \quad & G(x) \in \mathcal{K}
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This is a very broad framework including many important problems: LP, NLP, SDP, NLSDP, MOP, ...
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The Lagrangian function \( L : \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R} \) is defined by

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The **Karush-Kuhn-Tucker (KKT)** optimality condition for perturbed problem takes the following form:

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  a = \nabla_x L(x; y), \\
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GE:

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- \(M(x, a, b)\): the set of **Lagrange multipliers** associated with \((x, a, b)\), i.e.,
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- Aubin property
Calmness (Robinson’s upper Lipschitzian)

**Definition**

The set-valued mapping $\Psi$ is said to be calm at $\bar{p}$ if there exist a constant $\kappa > 0$ and an open neighborhood $\mathcal{U}$ of $\bar{p}$ such that

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\Psi(p) \subset \Psi(\bar{p}) + \kappa \| p - \bar{p} \| \mathbb{B} \quad \forall \, p \in \mathcal{U}.
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- It was called “upper Lipschitzian” by Robinson (1979)$^1$.

# (Robust) Isolated calmness

## Definition

The set-valued mapping $\Psi$ is said to be **isolated calm** at $\bar{p}$ for $\bar{q}$ if there exist a constant $\kappa > 0$ and open neighborhoods $\mathcal{U}$ of $\bar{p}$ and $\mathcal{V}$ of $\bar{q}$ such that

$$\Psi(p) \cap \mathcal{V} \subset \{\bar{q}\} + \kappa\|p - \bar{p}\|_B \quad \forall p \in \mathcal{U}. \quad (1)$$

Moreover, $\Psi$ is said to be **robustly isolated calm** at $\bar{p}$ for $\bar{q}$ if (1) holds and for each $p \in \mathcal{U}$, $\Psi(p) \cap \mathcal{V} \neq \emptyset$. 
• Isolated calmness: the "local upper Lipschitz continuity" Dontchev & Rockafellar (1997)\(^2\) and Levy (1996)\(^3\)

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In general, isolated calmness ≠ robust isolated calm

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• **Isolated calmness**: the “local upper Lipschitz continuity” \cite{DontchevRockafellar1997} and \cite{Levy1996}

• In general, isolated calmness \(\not\Rightarrow\) robust isolated calm

  a counterexample: Example 6.4 in \cite{MordukhovichOutrataRamirez2015}

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• In general, isolated calmness $\not\Rightarrow$ robust isolated calm

• **Robust isolated calm** = isolated calm + lower semi-continuous

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Aubin property

Definition

The set-valued mapping $\Psi$ has the **Aubin property** at $\bar{p}$ for $\bar{q}$ if there exist a constant $\kappa > 0$ and open neighborhoods $\mathcal{U}$ of $\bar{p}$ and $\mathcal{V}$ of $\bar{q}$ such that

$$\Psi(p) \cap \mathcal{V} \subset \Psi(p') + \kappa \| p - p' \|_{\mathbb{B}} \quad \forall \ p, p' \in \mathcal{U}.$$
**Aubin property**

<table>
<thead>
<tr>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>The set-valued mapping $\Psi$ has the <strong>Aubin property</strong> at $\bar{p}$ for $\bar{q}$ if there exist a constant $\kappa &gt; 0$ and open neighborhoods $\mathcal{U}$ of $\bar{p}$ and $\mathcal{V}$ of $\bar{q}$ such that</td>
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- It was designated “pseudo-Lipschitzian” by Aubin (1984)$^5$.
- Aubin property + “single-valuedness” = Robinson’s strong regularity$^6$

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Characterize the robust isolated calmness of $S_{\text{KKT}}$ for a class of non-polyhedral conic programming problems.
Why is it important?

A possible answer: It is related to the convergence analysis of numerical algorithms for solving OPs.
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Proximal point algorithm (PPA)

\( \mathcal{T} \): a maximal monotone operator from \( \mathcal{X} \) to \( \mathcal{X} \)

Solve the following inclusion problem

\[ 0 \in \mathcal{T}(z) \]

Given \( c > 0 \), the proximal mapping associated with \( c \mathcal{T} \):

\[ P := (I + c \mathcal{T})^{-1} \]

The proximal point algorithm (PPA):

\[ z_{k+1} \approx P_k(z_k) \]

Criteria for approximate calculation of \( P_k(z_k) \):

\[(A): \| z_{k+1} - P_k(z_k) \| \leq \delta_k \| z_{k+1} - z_k \|, \quad \infty \sum_{k=0}^\infty \delta_k < \infty \]
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Theorem (Rockafellar 1976\textsuperscript{7})

Let $z^k$ be generated by PPA using criterion (A) with $c_k$ nondecreasing ($c_k \uparrow c_\infty \leq \infty$). Suppose that $T^{-1}$ is \textit{robustly isolated calm} at 0 with modulus $\kappa$. Then,

- $z^k \to \bar{z}$ linearly with a rate bounded from above by
  \[
  \frac{\kappa}{\sqrt{\kappa^2 + c_\infty^2}} < 1 \quad \text{(fast linear)}
  \]

- If $c_\infty = \infty$, the convergence is \textit{superlinear}.

Outline

Background: the polyhedral case

Main results
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For example, for NLP:

- **Dontchev & Rockafellar (1997):** at a locally optimal solution,

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S_{\text{KKT}} \text{ is robustly isolated calm } \iff \left\{ \begin{array}{c}
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**Question:**

What about the non-polyhedral case, e.g., $\mathcal{K} = S^n_+$?

---


An example

Example 4.54 in Bonnans & Shapiro (2000):\(^{10}\)

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\begin{align*}
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- if \( \varepsilon = 0 \): a convex quadratic SDP problem with a strongly convex objective function and with the Slater condition being satisfied
- the unique optimal solution \( \bar{x} = (0, 0) \) with the unique Lagrange multiplier \( \bar{Y} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \).

For any given $\varepsilon \geq 0$, the perturbed problem has a unique optimal solution
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Robust isolated calmness $\iff$ \begin{align*}
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\text{SOSC} &
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Metric projection operator $\Pi_K$:

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If $K$ is a polyhedral closed convex set,
Polyhedral $\implies$ non-polyhedral

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If $K$ is a polyhedral closed convex set,

- $\Pi_K$ is directional differentiable \textbf{Facchinei & Pang} (2003)$^{12}$

$$\Pi_K(C + H) - \Pi_K(C) = \Pi_{C_K(C)}(H) =: \Pi'_K(C; H) \quad \forall \, H$$

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- $C_K(C)$ is the critical cone of $K$ at $C$

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- $\Pi_{\mathcal{K}}$ is directional differentiable and $\Pi'_{\mathcal{K}}(C; H)$ is the unique optimal solution to Bonnans et al. (1998)\textsuperscript{13}:

\[ \min \left\{ \|D - H\|^2 - \sigma(B, T_{\mathcal{K}}^2(\overline{A}, D)) \mid D \in C_{\mathcal{K}}(C) \right\} \]

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- $\overline{B} := C - \overline{A}$ and $\sigma(\overline{B}, T^2_{\mathcal{K}}(\overline{A}, D))$ is the “sigma” term of $\mathcal{K}$, cf. e.g., Bonnans and Shapiro (2000).

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C²-cone reducibility

Definition

The closed convex set \( \mathcal{K} \) is said to be **C²-cone reducible** at \( \overline{A} \in \mathcal{K} \), if there exist a open neighborhood \( \mathcal{W} \subset \mathcal{Y} \) of \( \overline{A} \), a pointed closed convex cone \( \mathcal{Q} \) (a cone is said to be pointed if and only if its lineality space is the origin) in a finite dimensional space \( \mathcal{Z} \) and a twice continuously differentiable mapping \( \Xi : \mathcal{W} \to \mathcal{Z} \) such that: (i) \( \Xi(\overline{A}) = 0 \in \mathcal{Z} \); (ii) the derivative mapping \( \Xi'(\overline{A}) : \mathcal{Y} \to \mathcal{Z} \) is onto; (iii) \( \mathcal{K} \cap \mathcal{W} = \{ A \in \mathcal{W} \mid \Xi(A) \in \mathcal{Q} \} \). We say that \( \mathcal{K} \) is C²-cone reducible if \( \mathcal{K} \) is C²-cone reducible at every \( \overline{A} \in \mathcal{K} \).
## $C^2$-cone reducibility

### Definition

The closed convex set $\mathcal{K}$ is said to be **$C^2$-cone reducible** at $\overline{A} \in \mathcal{K}$, if there exist a open neighborhood $\mathcal{W} \subset \mathcal{Y}$ of $\overline{A}$, a pointed closed convex cone $Q$ (a cone is said to be pointed if and only if its lineality space is the origin) in a finite dimensional space $\mathcal{Z}$ and a twice continuously differentiable mapping $\Xi : \mathcal{W} \rightarrow \mathcal{Z}$ such that:

1. $\Xi(\overline{A}) = 0 \in \mathcal{Z}$;
2. the derivative mapping $\Xi'(\overline{A}) : \mathcal{Y} \rightarrow \mathcal{Z}$ is onto;
3. $\mathcal{K} \cap \mathcal{W} = \{ A \in \mathcal{W} | \Xi(A) \in Q \}$. We say that $\mathcal{K}$ is $C^2$-cone reducible if $\mathcal{K}$ is $C^2$-cone reducible at every $\overline{A} \in \mathcal{K}$.

- a closed polyhedral convex set; SOC; PSD cone; ...

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(Y. Cui, C. Ding and X.Y. Zhao, Quadratic Growth Conditions for Convex Matrix Optimization Problems Associated with Spectral Functions, to appear in SIAM Journal on Optimization (2017).)
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- **spectral functions:** nuclear norm; Ky Fan $k$-norm; ...

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- a closed polyhedral convex set; SOC; PSD cone; ...
- spectral functions: nuclear norm; Ky Fan $k$-norm; ... (cf. Cui et al. (2017)\textsuperscript{14} for more details)

Main results
Robust isolated calm = isolated calm + lower semi-continuous
The lower semi-continuity of $S_{\text{KKT}}$

<table>
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<th>Proposition</th>
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<td>Suppose that $\bar{x}$ is an isolated locally optimal solution with $(a, b) = (0, 0)$ and the corresponding set of Lagrange multipliers $M(\bar{x}, 0, 0) \neq \emptyset$. If the strict Robinson CQ holds at $\bar{x}$ with respect to $\bar{y} \in M(\bar{x}, 0, 0)$, then the KKT solution mapping $S_{\text{KKT}}$ is lower semi-continuous at $(0, 0, \bar{x}, \bar{y}) \in \text{gph } S_{\text{KKT}}$.</td>
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The lower semi-continuity of $S_{\text{KKT}}$

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- The strict Robinson CQ (SRCQ) is said to hold with $(a, b) = (0, 0)$ at $\bar{x}$ with respect to $\bar{y} \in M(\bar{x}, 0, 0) \neq \emptyset$ if

  $$G'(\bar{x})\mathcal{X} + \mathcal{T}_K(G(\bar{x})) \cap \bar{y}^\perp = \mathcal{Y}.$$
The lower semi-continuity of $S_{KKT}$

**Proposition**

Suppose that $\bar{x}$ is an isolated locally optimal solution with $(a, b) = (0, 0)$ and the corresponding set of Lagrange multipliers $M(\bar{x}, 0, 0) \neq \emptyset$. If the **strict Robinson CQ** holds at $\bar{x}$ with respect to $\bar{y} \in M(\bar{x}, 0, 0)$, then the KKT solution mapping $S_{KKT}$ is **lower semi-continuous** at $(0, 0, \bar{x}, \bar{y}) \in \text{gph } S_{KKT}$.

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- The set of Lagrange multipliers $M(\bar{x}, 0, 0)$ is a **singleton** if the SRCQ holds.
The isolatedness of $X_{\text{KKT}}$

We can extend Robinson’s classical result on the isolatedness of an optimal solution Robinson (1982) to the non-polyhedral case by the reduction approach.
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**Proposition**

Suppose that the RCQ holds at a locally optimal solution $\bar{x}$ with $(a, b) = (0, 0)$ and that Robinson’s SOSC

$$\inf_{y \in M(\bar{x}, 0, 0)} \left\{ \langle d, \nabla_{xx}^2 L(\bar{x}; y) d \rangle - \sigma \left( y, T^2_K(G(\bar{x}), G'(\bar{x})d) \right) \right\} > 0 \quad \forall \ d \in C(\bar{x}) \setminus \{0\}$$

holds at $\bar{x}$. Then, there exists an open neighborhood $V$ of $\bar{x}$ such that $X_{\text{KKT}}(0, 0) \cap V = \{x\}$, which implies that $\bar{x}$ is an isolated locally optimal solution with $(a, b) = (0, 0)$. 


The equivalent reformulation

When \((a, b) = (0, 0)\), the KKT system is equivalent to the following system of non-smooth equations:

\[ F(x, y) = 0, \]

where \(F : \mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{X} \times \mathcal{Y} \) is the natural mapping defined by

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F(x, y) := \begin{bmatrix}
\nabla f(x) + G'(x)^* y \\
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**Lemma**

Let \((0, 0, \bar{x}, \bar{y}) \in \text{gph} \ S_{\text{KKT}}\). The set-valued mapping \(S_{\text{KKT}}\) is isolated calm at the origin for \((\bar{x}, \bar{y})\) if and only if the set-valued mapping \(F^{-1}\) is isolated calm at the origin for \((\bar{x}, \bar{y})\).
The characterization of the robust isolated calmness

<table>
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<th>Theorem</th>
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Let $\bar{x}$ be a feasible solution with $(a, b) = (0, 0)$. Suppose that the RCQ holds at $\bar{x}$. Assume that $\mathcal{K}$ is $C^2$-cone reducible at $G(\bar{x})$ with respect to $\bar{y} \in M(\bar{x}, 0, 0) \neq \emptyset$. Then the following statements are equivalent:

(i) the SRCQ holds at $\bar{x}$ with respect to $\bar{y}$ and the SOSC holds at $\bar{x}$ with $(a, b) = (0, 0)$;

(ii) $\bar{x}$ is a locally optimal solution with $(a, b) = (0, 0)$ and $S_{\text{KKT}}$ is **robustly isolated calm** at the origin for $(\bar{x}, \bar{y})$;

(iii) $\bar{x}$ is a locally optimal solution with $(a, b) = (0, 0)$ and $S_{\text{KKT}}$ is **isolated calm** at the origin for $(\bar{x}, \bar{y})$.  


The isolated calmness of the mapping $F^{-1}$ at the origin for $(\bar{x}, \bar{y})$ implies the following error bound result: there exist a constant $\kappa > 0$ and a neighborhood $\mathcal{V}$ of $(\bar{x}, \bar{y})$ in $\mathcal{X} \times \mathcal{Y}$ such that

$$\| (x, y) - (\bar{x}, \bar{y}) \| \leq \kappa \| F(x, y) \| \quad \forall (x, y) \in \mathcal{V}. $$
By combining Fusek (2013), Klatte and Kummer (2013) and Fusek (2001), we obtain

Proposition

Let $\bar{x}$ be a stationary point with $(a, b) = (0, 0)$. Suppose that $S_{KKT}$ has the Aubin property at the origin for $(\bar{x}, \bar{y})$ with $\bar{y} \in M(\bar{x}, 0, 0) \neq \emptyset$, then

- the constraint non-degeneracy condition holds at $\bar{x}$;
- $F^{-1}$ is isolated calm at the origin for $(\bar{x}, \bar{y})$.

The constraint non-degeneracy is said to hold with $(a, b) = (0, 0)$ at $\bar{x}$ if $G'(\bar{x}) X + \text{lin}\{T_{K}(G(\bar{x}))\} = Y$.

The constraint non-degeneracy is stronger than the SRCQ.
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Robust isolated calmness v.s. Aubin property

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\begin{align*}
\min & \quad \frac{1}{2} (X_{11} - 1)^2 + \frac{1}{2} (X_{22} - 2X_{12})^2 \\
\text{s.t.} & \quad \langle E, X \rangle \leq 1, \\
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- \( \overline{X} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \) is the unique optimal solution and 
  \((\overline{s}, \overline{Y}) = (0, 0) \in \mathbb{R} \times S^2 \) is the unique corresponding Lagrange multiplier.
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- Aubin property/strong regularity of \( S_{KKT} \) fails to hold since the strong SOSC does not hold.
Conclusions

Robust isolated calmness

Aubin property

Strong Regularity

SOSC + SRCQ

SOSC + Non-degeneracy

SOSC + Non-degeneracy

Robust isolated calmness

Convex

Strong Regularity

SOSC + SRCQ
Thank you