Matrix optimization:

recent progress on algorithm foundation

Chao Ding
Institute of Applied Mathematics
Academy of Mathematics and Systems Science
Chinese Academy of Sciences

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中国科学院
CHINESE ACADEMY OF SCIENCES

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- Y. Cui, C. Ding and X.Y. Zhao, *Quadratic Growth Conditions for Convex Matrix Optimization Problems Associated with Spectral Functions*, **SIAM Journal on Optimization** 27, 2332–2355 (2017).
- Y. Cui and C. Ding, *Nonsmooth Composite Matrix Optimizations: Strong Regularity, Constraint Nondegeneracy and Beyond*, **preprint** (2018).

USC University of Southern California

BISEC
BEIJING INSTITUTE FOR SCIENTIFIC AND ENGINEERING COMPUTING
北京科学与工程计算研究院

Outline

Matrix optimization problem

Augmented Lagrangian method

- **The convex case**
- **The non-convex case**

Two perturbation results

- **The calmness of X**
- **The strong regularity of the general MOP**

Matrix optimization problem

Matrix Optimization Problem (MOP)

MOP:

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- $\mathcal{Q}$: a convex polyhedral set

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Matrix completion:

$$\min \left\{ \|P_\Omega(X) - P_\Omega(M)\|_F^2 + \lambda \|X\|_* \mid \mathcal{A}(X) = b,\ X \in \mathcal{P} \right\}$$

— $P_\Omega(\cdot)$: the observation operator

More applications

- SDP
- Fastest mixing Markov chain problem (fast load balancing of paralleled systems)
- Fastest distributed linear averaging problem
- The reduced rank approximations of transition matrices
- The low rank approximations of doubly stochastic matrices
- Low-rank approximation of matrices with linear structures
- ......

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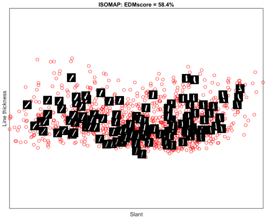
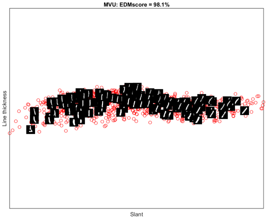
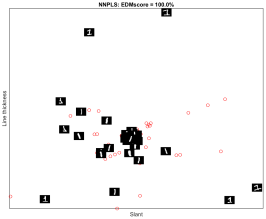
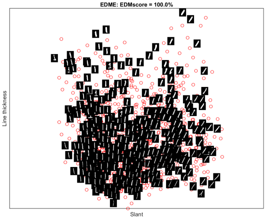
- **Unsupervised learning**

MOPs in unsupervised learning









MNIST dataset. Source: **D. and Qi, (2017)**

Solving MOPs: an example

The fastest mixing Markov chain problem:[1]

- find the edge transition probabilities that give the fastest mixing Markov chain
- equivalent to minimize the **second largest** eigenvalue modulus of P

$$\begin{array}{ll} \min & \|P\|_{(2)} \\ \text{s.t.} & P \geq 0, \quad P\mathbf{1} = \mathbf{1}, \quad P = P^T \\ & P_{ij} = 0, \quad (i,j) \notin \mathcal{E} \end{array}$$

[1] S. Boyd, P. Diaconis, and L. Xiao, SIAM Review, 46 (2004), pp. 667–689.

Numerical results

Table 1: The performance of the **augmented Lagrangian method** (**ALM**) and the ADMM for solving the FMMC problems. The computational time is in the format of "hh:mm:ss".

		iteration	η	time
problem	d; n	ALM | ADMM	ALM| ADMM	ALM | ADMM
cage	2562 ; 366	6;6;200 | 1925	0.0-7 | 9.1-7	05 | 37
G3	19176 ; 800	32;57;88 | 599	3.0-7 | 8.3-7	1:37 | 1:21
G6	9665 ; 800	30;44;145 | 989	8.5-7 | 9.6-7	1:09 | 1:52
G15	4661 ; 800	31;51;200 | 6122	3.4-8 | 7.7-7	1:05 | 11:27
G46	9990 ; 1000	30;44;134 | 1619	5.6-7 | 9.9-7	1:34 | 5:33
G54	5916 ; 1000	22;62;200 | 8928	7.7-7 | 9.9-7	2:23 | 27:16
G43	9990 ; 1000	24;96;90 | 2073	2.9-7 | 9.3-7	2:37 | 6:03
delaunayn10	3056 ; 1024	61;359;200 | 25000	6.8-9 | 7.2-5	10:25 | 1:04:29
G22	19990 ; 2000	31;46;56 | 2918	2.3-8 | 9.9-7	5:57 | 41:31
G24	19990 ; 2000	41;296;200 | 6808	2.7-7 | 9.9-7	53:57 | 1:38:37
G26	19990 ; 2000	29;87;200 | 2954	1.4-7 | 9.9-7	16:15 | 42:18
minnesota	3303 ; 2642	25;24;123 | 258	0.0-10 | 9.1-7	6:27 | 6:29
G48	6000 ; 3000	40;79;200 | 9470	9.2-7 | 9.2-7	19:39 | 4:40:08
G49	6000 ; 3000	25;38;200 | 7488	6.6-7 | 8.5-7	10:58 | 3:36:58
G50	6000 ; 3000	26;42;74 | 6370	4.8-8 | 7.8-7	9:36 | 2:59:34
USpowerGrid	6594 ; 4941	27;120;200 | 25000	1.4-7 | 1.0-5	3:11:22 | 56:47:18

Why ALM?

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ALM has the **asymptotically superlinear** convergence / **linearly convergent** of an **arbitrary** order

Perturbed MOPs

Canonically perturbed MOPs with parameters (a,b,c):

$$\begin{aligned} \text{min} \quad & f(x) - \langle \mathbf{a}, x \rangle + \theta(g(x) + \mathbf{b}) \\ \text{s.t.} \quad & h(x) + \mathbf{c} \in \mathcal{Q} \end{aligned}$$

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Under suitable CQs, the Karush-Kuhn-Tucker (KKT) optimality condition for perturbed problem takes the following form:

$$\begin{cases} \mathbf{a} = \nabla f(x) + g'(x)^* y + h'(x)^* z \\ y \in \partial\theta(g(x) + \mathbf{b}) \\ z \in \mathcal{N}_{\mathcal{K}}(h(x) + \mathbf{c}) \end{cases} \quad \Longleftrightarrow \quad \textbf{GE}: \quad (a, b, c) \in \mathcal{T}_L(x, y, z)$$

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$X(a, b, c)$: the set of all locally optimal solutions

$S_{\mathrm{KKT}}(a, b, c)$: the set of all solutions (x, y, z) to the KKT system

Perturbed MOPs (cont'd)

Let Ψ be a **set-valued mapping** with $(\overline{X}, \overline{Y}) \in \operatorname{gph} \Psi$. **Lipschitz-like** properties:

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  there exist neighborhoods U of $\overline{X}$, V of $\overline{Y}$ and constant $\kappa > 0$ such that
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Strong regularity:

$S_{\mathrm{KKT}}(a, b, c)$ is locally Lipschitz continuous.

Why it matters?

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- **Theory**: perturbation analysis of OPs

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- **Algorithm**: e.g., convergence rate of augmented Lagrangian method

Augmented Lagrangian method

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  - global convergence
  - local (super)linear convergence

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- Non-convex case: for NLP, under suitable conditions, local convergence at a linear rate
  Powell (1972); **Rockafellar (1973a), (1973b)**; **Tretyakov (1973)**, **Bertsekas (1976)**; **Conn et al. (1991)**, **Contesse-Back (1993)**; **Ito and Kunisch (1990)**; ...

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A survey paper:

R.T. Rockafellar. Lagrange multipliers and optimality. *SIAM Review*, 35: 183–238, 1993.

Augmented Lagrangian method

The convex case

The non-convex case

The convex case

Consider the following quadratic MOP and its dual:

$$\begin{array}{lll} (P) & \min & \frac{1}{2}\|\mathcal{F}X - d\|^2 + \langle C, X\rangle + \theta(X) \\ & \text{s.t.} & \mathcal{A}X = b,\ X \in \mathcal{P} \end{array}$$

$$\begin{array}{lll} (D) & \min & \frac{1}{2}\|w\|^2 - \langle w, d\rangle - \langle b, y\rangle + \theta^*(-S) + \delta_P^*(-Z) \\ & \text{s.t.} & \mathcal{F}^* w + \mathcal{A}^* y + S + Z = C \end{array}$$

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The **Lagrange function** of (D):

$$\begin{aligned}L(w, y, S, Z, X) := \quad & \tfrac{1}{2}\|w\|^2 - \langle b, y\rangle - \langle d, w\rangle + \theta^*(-S) + \delta_P^*(-Z) \\ & + \langle X, \mathcal{F}^* w + \mathcal{A}^* y + S + Z - C\rangle,\end{aligned}$$

ALM (cont'd)

The **augmented Lagrangian function** associated with (D)

$$L_\sigma(w, y, S, Z, X) := L(w, y, S, Z, X) + \frac{\sigma}{2}\|\mathcal{F}^* w + \mathcal{A}^* y + S + Z - C\|^2.$$

ALM for solving (D):

Given a sequence of scalars $\sigma_k \uparrow \sigma_\infty \leq \infty$ and a starting point $X^0 \in \mathbb{R}^{m\times n}$, the $(k+1)$-th iteration of the **inexact** ALM:

$$\begin{cases} z^{k+1} := (w^{k+1}, y^{k+1}, S^{k+1}, Z^{k+1}) \approx \arg\min \zeta_k(z) := L_{\sigma_k}(w, y, S, Z, X^k) \\ X^{k+1} = X^k + \sigma_k(\mathcal{F}^* w^{k+1} + \mathcal{A}^* y^{k+1} + S^{k+1} + Z^{k+1} - C) \end{cases}$$

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Terminate the subproblem for solving z^{k+1} by one of following criteria:

$$(A) \quad \zeta_k(z^{k+1}) - \inf \zeta_k(z) \leq \varepsilon_k^2/2\sigma_k, \ \varepsilon_k \geq 0, \ \sum_{k=0}^{\infty} \varepsilon_k < \infty$$

$$(B) \quad \zeta_k(z^{k+1}) - \inf \zeta_k(z) \leq (\eta_k^2/2\sigma_k)\|X^{k+1} - X^k\|^2, \ \eta_k \geq 0, \ \sum_{k=0}^{\infty} \eta_k < \infty$$

ALM & PPA

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$F(X) := \inf \left\{ L(w, y, S, Z, X) \mid (w, y, S, Z) \in \Re^d \times \Re^e \times \Re^{m\times n} \times \Re^{m\times n} \right\}$

$\mathcal{T}_{\mathbf{F}} := \partial F$ — a maximal monotone operator

Optimal solution set of (P): $X(0) = \{X \in \mathcal{X} \mid 0 \in \mathcal{T}_F(X)\}$

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Given $\sigma_k > 0$, the **proximal mapping** associated with $\sigma_k \mathcal{T}_F$:

$$\mathcal{P} := (\mathcal{I} + \sigma_k \mathcal{T}_F)^{-1}$$

The **proximal point algorithm** (PPA):

$$X^{k+1} \approx \mathcal{P}_k(X^k), \quad \mathcal{P}_k := (\mathcal{I} + \sigma_k \mathcal{T}_F)^{-1}$$

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Criteria for approximate calculation of $\mathcal{P}_k(X^k)$:

$$(A) \quad \|X^{k+1} - \mathcal{P}_k(X^k)\| \le \varepsilon_k^2 / 2\sigma_k$$

$$(B) \quad \|X^{k+1} - \mathcal{P}_k(X^k)\| \le (\eta_k^2 / 2\sigma_k) \|X^{k+1} - X^k\|^2$$

Convergent rate

Rockafellar (1976)[2] established the convergence rate under the **robust isolated calmness**

[2] R.T. Rockafellar. *SIAM Journal on Control and Optimization*, 14(5): 877–898, 1976.

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- allow the multi-solution case and is satisfied when $\mathcal{T}_F$ is **polyhedral** (e.g., LASSO)

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- allow the multi-solution case and is satisfied when $\mathcal{T}_F$ is **polyhedral** (e.g., LASSO)
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[2] R.T. Rockafellar. *SIAM Journal on Control and Optimization*, 14(5): 877–898, 1976.
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Convergent rate

Rockafellar (1976)[2] established the convergence rate under the **robust isolated calmness**

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> **Question**: For MOPs, how to ensure the convergence of ALM and how fast it can be?

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Convergence rate of ALM

Theorem (asymptotic superlinear for ALM)

Assume that $\mathcal{T}_F^{-1}(0) \neq \emptyset$. *(i) Let* $\{X^k\}$ *be a sequence generated by the PPA with stopping criterion* (A). *Then* $\{X^k\}$ *converges to some* $X^\infty \in X(0)$. *(ii) If the criterion (B) is also employed and optimal solution mapping* $X = \mathcal{T}_F^{-1}$ *is* **calm** *at the origin for* X^∞ *with modulus* $\kappa_p \geq 0$, *then there exists* $\bar{k} \geq 0$ *such that for all* $k \geq \bar{k}$, $\eta_k < 1$ *and*

$$\operatorname{dist}(X^{k+1}, \mathcal{T}_F^{-1}(0)) \leq \theta_k \operatorname{dist}(X^k, \mathcal{T}_F^{-1}(0)),$$

where

$$\theta_k = (\mu_k + 2\eta_k)(1-\eta_k)^{-1} \quad \textit{with} \quad \mu_k = \kappa_p / \sqrt{\kappa_p^2 + \sigma_k^2}\ ,$$

$$\theta_k \to \theta_\infty = \kappa_p / \sqrt{\kappa_p^2 + \sigma_\infty^2} \quad (\theta_\infty = 0 \text{ if } \sigma_\infty = \infty).$$

$\|\mathcal{F}^* w^{k+1} + \mathcal{A}^* y^{k+1} + S^{k+1} + Z^{k+1} - C\| \leq \tau_k^1 \operatorname{dist}(X^k, \mathcal{T}_F^{-1}(0))$ *and* $\Psi(z^{k+1}) - \Psi^* \leq \tau_k^2 \operatorname{dist}(X^k, \mathcal{T}_F^{-1}(0))$. *where* $\tau_k^1 \to \tau_\infty^1 = 1/\sigma_\infty$, $\tau_k^2 \to \tau_\infty^2 = \|X^\infty\|/\sigma_\infty$ *(*$\tau_\infty^1 = \tau_\infty^2 = 0$ if $\sigma_\infty = \infty$*).*

Convergence rate of ALM (cont'd)

Roughly speaking, $X^k \to X^\infty$ **linearly** with a rate bounded from above by

$$\frac{\kappa_p}{\sqrt{\kappa_p^2 + \sigma_\infty^2}} < 1 \quad \textbf{(fast linear)}$$

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$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & \phi(x) = 0 \end{array} \qquad \textbf{ALM}: \quad \begin{cases} x^{k+1} = \mathrm{argmin}_x \; L_\sigma(x; y) \\ y^{k+1} = y^k + \sigma\phi(x^{k+1}) \end{cases}$$

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$y^{k+1} = y^k + \sigma\phi(x^{k+1}) \Leftarrow$ **gradient method** for $\Psi_\sigma(y) := \phi(x(y, \sigma)) = 0$

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Powell (1969) shows that the Jacobian $-J_\sigma^*$ of Ψ_σ^* satisfies

$$\|-J_\sigma^* - I\| = O(\frac{1}{\sigma - c})$$

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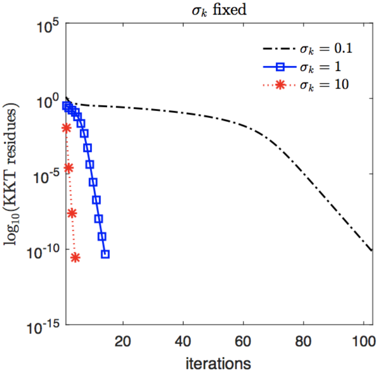
Convergence rate of ALM (cont'd)



Source: **Cui, Sun and Toh (2018)**

Augmented Lagrangian method

The convex case

The non-convex case

The non-convex case

- For the nonlinear SDP problem, **Sun, Sun and Zhang (2007)**[5]: ALM converges linearly under the **strong regularity**, i.e., $S_{\rm KKT}$ is locally Lipschitz continuous

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- For the nonlinear SDP problem, **Sun, Sun and Zhang (2007)**[5]: ALM converges linearly under the **strong regularity**, i.e., S_{KKT} is locally Lipschitz continuous
- For the general conic optimization, **Kanzow and Steck (2018)**[6]: a modified ALM converges linearly under the **robust isolated calmness** of S_{KKT}

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- A characterization for S_{KKT} in general **non-polyhedral** and **non-convex** cases is provided by **D., Sun and Zhang (2017)**[7]

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[7] C. Ding, D.F. Sun and L.W. Zhang. *SIAM Journal on Optimization*, 27: 67-90, 2017.

Two perturbation results

Calmness and strong regularity

- The **calmness** of the optimal solution X for the convex MOPs, i.e., there exist neighborhoods U of $\bar{u}$, V of $\overline{X} \in X(\bar{u})$ and constant $\kappa > 0$ such that

$$X(u) \cap V \subseteq X(\bar{u}) + \kappa \|u - \bar{u}\| \mathbb{B} \quad \forall\, u \in U.$$

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- The **strong regularity** of the KKT solution S_{KKT} for the general non-convex MOPs, i.e.,

S_{KKT} is a local Lipschitz continuous function.

Two perturbation results

The calmness of X

The strong regularity of the general MOP

The convex MOPs

Recall the following quadratic MOP and its dual:

$$(P)\quad \min \quad \tfrac{1}{2}\|\mathcal{F}X - d\|^2 + \langle C, X\rangle + \theta(X)$$
$$\text{s.t.} \quad \mathcal{A}X = b,\ X \in \mathcal{Q}$$

$$(D)\quad \min \quad \tfrac{1}{2}\|w\|^2 - \langle w, d\rangle - \langle b, y\rangle + \theta^*(-S) + \delta^*_{\mathcal{Q}}(-Z)$$
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The **Lagrange function** of (D):

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The **essential objective function** of (P)

$F(X) := \inf \left\{ L(w, y, S, Z, X) \mid (w, y, S, Z) \in \Re^d \times \Re^e \times \Re^{m\times n} \times \Re^{m\times n} \right\}$

$\mathcal{T}_{\mathbf{F}} := \partial F$ — a maximal monotone operator

Optimal solution set of (P): $X(0) = \{X \in \mathcal{X} \mid 0 \in \mathcal{T}_F(X)\}$

The calmness of X

Theorem (sufficient conditions for calmness)

$X = \mathcal{T}_F^{-1}$ is **calm** *at 0 for $\overline{X}$ if one of the following two conditions holds:*

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(i) the function θ is a $\mathcal{C}^2$-cone reducible function and the second order sufficient condition of (P) holds at $\overline{X}$.

*(ii) for any $v \in \partial\theta(X)$, $(\partial\theta)^{-1}$ is **calm** at v for X and there exists an optimal solution $\widehat{X}$ and Lagrange multipliers $(\hat{w}, \hat{y}, \widehat{S}, \widehat{Z})$ such that the strict complementarity holds at $(\widehat{X}, \hat{w}, \hat{y}, \widehat{S}, \widehat{Z})$.*

The calmness of $(\partial\theta)^{-1}$

Recall $\theta(X) = \phi(\sigma(X)) \; \forall \; X \in \mathbb{R}^{m\times n}$.

Theorem

Let $\phi : \mathbb{R}^m \to (-\infty, +\infty]$ be an **absolutely symmetric function**. *For any $(\overline{X}, \overline{W}) \in$ gph $\partial\theta$, if the inverse of the subdifferential mapping $(\partial\phi)^{-1}$ is calm at $\sigma(\overline{W})$ for $\sigma(\overline{X})$, then $(\partial\theta)^{-1}$ is calm at $\overline{W}$ for $\overline{X}$.*

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- $\partial\phi$ is a **polyhedral mapping**, **S.M. Robinson** (1981)
- ϕ is a **piecewise linear-quadratic function**, **J. Sun**'s Ph.D thesis (1986)

  e.g., $\delta_{\mathcal{S}^n_+}$; $\delta_{\rm EDM}$; $\|\cdot\|_2$; $\|\cdot\|_*$; $\|\cdot\|_{(k)}$; ...

Two perturbation results

The calmness of X

The strong regularity of the general MOP

The general MOPs

MOP:

$$\begin{aligned} \min_{x \in \mathbb{X}} \quad & f(x) + \theta(g(x)) \\ \text{s.t.} \quad & h(x) \in \mathcal{Q} \end{aligned}$$

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- $\theta = \phi \circ \sigma$: a function defined on the finite dimensional matrix space
- $\phi : \mathbb{R}^m \to (-\infty, \infty]$ is a convex piecewise linear function:

$$\phi(x) = \phi_1(x) + \phi_2(x), \quad x \in \mathbb{R}^m$$

with

$$\phi_1(x) := \max\{\langle \eta_1, x\rangle - \xi_1, \ldots, \langle \eta_l, x\rangle - \xi_l\}$$

and $\phi_2(x) := \delta_{\mathrm{dom}\ \phi}(x)$ where

$$\mathrm{dom}\ \phi := \{x \mid \langle \gamma_i, x\rangle - \iota_i \le 0,\ i = 1, \ldots, r\}$$

Second-order optimality condition

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The "no-gap" **(strong) second order sufficient condition** holds at $\overline{X}$, if for any $d \in \mathcal{C}(\bar{x}) \setminus \{0\}$ ($d \in \widehat{\mathcal{C}}(\bar{x}) \setminus \{0\}$),

$$\sup_{(\bar{y},\overline{S}) \in \mathcal{M}(\bar{x})} \left\{ \langle d, L''_{xx}(\bar{x}, \bar{y}, \overline{S}) d \rangle - \Upsilon_{g(\bar{x})} \left(\overline{S}, g'(\bar{x}) d \right) \right\} > 0$$

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- $\Upsilon_{g(\bar{x})}\left(\overline{S}, g'(\bar{x})d\right)$: the so-called "**$\sigma$-term**"

Constraint nondegeneracy

The **constraint nondegeneracy** for the MOP:

$$\left[\begin{array}{c} h'(\bar{x}) \\ g'(\bar{x}) \end{array}\right] \mathbb{X} + \left[\begin{array}{c} \mathrm{lin}\left(\mathcal{T}_{\mathcal{Q}}(h(\bar{x}))\right) \\ \mathcal{T}^{\mathrm{lin}}(g(\bar{x})) \end{array}\right] = \left[\begin{array}{c} \mathbb{R}^d \\ \mathbb{V} \end{array}\right],$$

where the linear space $\mathcal{T}^{\mathrm{lin}}$ is called the linearity space of the θ defined by

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Most importantly, the **explicit formulas** for above are now available for the case ϕ is a convex piecewise linear function **Cui and Ding (2018)**.

Characterization of the strong regularity for MOPs

Theorem

Let $\bar{x} \in \mathbb{X}$ be a feasible solution to the MOP with $\mathcal{M}(\bar{x}) \neq \emptyset$. Suppose that the Robinson constraint qualification (RCQ) holds at $\bar{x}$ and $(\bar{y}, \overline{S}) \in \mathcal{M}(\bar{x})$. Then, the following statements are equivalent:

(i) *The strong second order sufficient condition and constraint nondegeneracy for the MOP both hold at $\bar{x}$;*

(ii) $(\bar{x}, \bar{y}, \overline{S})$ *is a strongly regular solution of the KKT equations.*

Conclusions

MOPs in Data Science

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ALM for MOPs

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Sufficient conditions for the **calmness** of X

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MOPs in Data Science

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Characterization of the **strong regularity** of the general MOP

Thank you!