Characterization of the Robust Isolated Calmness for a Class of Conic Programming Problems

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This is a joint work with Defeng Sun at National University of Singapore and Liwei Zhang at Dalian University of Technology.

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This is a very broad framework including many important problems: LP, NLP, SDP, NLSDP, MOP, ...

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The Lagrangian function $L : \mathcal{X} \times \mathcal{Y} \to \Re$ is defined by

$$L(x; y) := f(x) + \langle y, G(x)\rangle, \quad (x, y) \in \mathcal{X} \times \mathcal{Y}$$

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The Karush-Kuhn-Tucker (KKT) optimality condition for perturbed problem takes the following form:

$$\left\{ \begin{array}{l} a = \nabla_x L(x; y), \\ b \in -G(x) + \partial\sigma(y, \mathcal{K}) \end{array} \right. \quad \Longleftrightarrow \quad \left\{ \begin{array}{l} a = \nabla_x L(x; y), \\ y \in \mathcal{N}_{\mathcal{K}}(G(x) + b) \end{array} \right.$$

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GE:

$$(a, b) \in \mathcal{T}_L(x, y)$$

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- $M(x, a, b)$: the set of Lagrange multipliers associated with (x, a, b), i.e.,

$$\{y \in \mathcal{Y} \mid (x, y) \in S_{\mathrm{KKT}}(a, b)\}$$

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- (robust) isolated calmness
- Aubin property

Calmness (Robinson's upper Lipschitzian)

Definition

The set-valued mapping Ψ is said to be calm at $\bar{p}$ if there exist a constant $\kappa > 0$ and an open neighborhood $\mathcal{U}$ of $\bar{p}$ such that

$$\Psi(p) \subset \Psi(\bar{p}) + \kappa \|p - \bar{p}\| \mathbb{B} \quad \forall\, p \in \mathcal{U}.$$

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- It was called "upper Lipschitzian" by **Robinson** (1979)[1].

[1] S.M. Robinson. Generalized equations and their solutions, Part I: Basic theory. *Mathematical Programming Study* 10 (1979) 128–141.

(Robust) Isolated calmness

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$$\Psi(p) \cap \mathcal{V} \subset \{\bar{q}\} + \kappa \|p - \bar{p}\| \mathbb{B} \quad \forall\, p \in \mathcal{U}. \tag{1}$$

Moreover, Ψ is said to be robustly isolated calm at $\bar{p}$ for $\bar{q}$ if (1) holds and for each $p \in \mathcal{U}$, $\Psi(p) \cap \mathcal{V} \neq \emptyset$.

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- Isolated calmness: the "local upper Lipschitz continuity" **Dontchev & Rockafellar** (1997)[2] and **Levy** (1996)[3]

[2] A.L. Dontchev and R.T. Rockafellar. Characterizations of Lipschitzian stability in nonlinear programming. in *Mathematical Programming With Data Perturbations*, A.V. Fiacco, ed.), Marcel Dekker, New York, 1997, 65–82.

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- Robust isolated calm $=$ isolated calm $+$ lower semi-continuous

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Aubin property

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- Aubin property + "single-valuedness" = Robinson's strong regularity[6]

[5] J.P. Aubin. Lipschtiz behavior of solutions to convex minimization problems. *Mathematics of Operations Research* 9 (1984) 87–111.

[6] S.M. Robinson. Strongly regular generalized equations. *Mathematics of Operations Research* 5 (1980) 43–62.

Main topic

Characterize the robust isolated calmness of S_{KKT} for a class of non-polyhedral conic programming problems

Why is it important?

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A possible answer: It is related to the convergence analysis of numerical algorithms for solving OPs.

Proximal point algorithm (PPA)

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Given $c > 0$, the proximal mapping associated with $c\mathcal{T}$:

$$\mathcal{P} := (\mathcal{I} + c\mathcal{T})^{-1}$$

The proximal point algorithm (PPA):

$$z^{k+1} \approx \mathcal{P}_k(z^k), \quad \mathcal{P}_k := (\mathcal{I} + c_k\mathcal{T})^{-1}$$

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Criteria for approximate calculation of $\mathcal{P}_k(z^k)$:

$$(\mathrm{A}): \quad \|z^{k+1} - \mathcal{P}_k(z^k)\| \le \delta_k \|z^{k+1} - z^k\|, \quad \sum_{k=0}^{\infty} \delta_k < \infty$$

Convergence rate of PPA

Theorem (Rockafellar 1976[7])

Let z^k be generated by PPA using criterion (A) with c_k nondecreasing ($c_k \uparrow c_\infty \leq \infty$). Suppose that $\mathcal{T}^{-1}$ is robustly isolated calm at 0 with modulus κ. Then,

- *$z^k \to \bar{z}$ linearly with a rate bounded from above by*

$$\frac{\kappa}{\sqrt{\kappa^2 + c_\infty^2}} < 1 \quad \textit{(fast linear)}$$

- *If $c_\infty = \infty$, the convergence is superlinear.*

[7] R.T. Rockafellar. Monotone operators and the proximal point algorithm. *SIAM Journal on Control and Optimization* 14 (1976) 877–898.

Outline

Background: the polyhedral case

Main results

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The polyhedral case

When the set $\mathcal{K}$ is polyhedral, the theory is fairly complete: **Robinson** (1982)[8], **Dontchev & Rockafellar** (1997), **Klatte & Kummer** (2002)[9], etc.

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For example, for NLP:

- **Dontchev & Rockafellar** (1997): at a locally optimal solution,

$$S_{\mathrm{KKT}} \text{ is robustly isolated calm} \iff \begin{cases} \text{strict MFCQ} \\ \text{second order sufficient condition} \end{cases}$$

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Question:

What about the non-polyhedral case, e.g., $\mathcal{K} = \mathcal{S}^n_+$?

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An example

Example 4.54 in **Bonnans & Shapiro** (2000)[10]:

$$\begin{array}{ll} \min & x_1 + x_1^2 + x_2^2 \\ \text{s.t.} & \mathrm{Diag}(x) + \varepsilon A \in \mathcal{S}_+^2 \end{array}$$

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- if $\varepsilon = 0$: a convex quadratic SDP problem with a strongly convex objective function and with the Slater condition being satisfied
- the unique optimal solution $\bar{x} = (0, 0)$ with the unique Lagrange multiplier $\overline{Y} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$.

[10] J.F. Bonnans and A. Shapiro. *Perturbation Analysis of Optimization Problems*, Springer, New York, 2000.

An example (cont'd)

For any given $\varepsilon \geq 0$, the perturbed problem has a unique optimal solution $X(\varepsilon) = (\bar{x}_1(\varepsilon), \bar{x}_2(\varepsilon))$ with $\bar{x}_2(\varepsilon)$ of order $\varepsilon^{2/3}$ as $\varepsilon \to 0$.

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Polyhedral $\Longrightarrow$ non-polyhedral

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If $\mathcal{K}$ is a polyhedral closed convex set,

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Polyhedral $\implies$ non-polyhedral (cont'd)

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Polyhedral $\Longrightarrow$ non-polyhedral (cont'd)

If $\mathcal{K}$ is a non-polyhedral closed convex set but C^2-cone reducible,

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- $\overline{B}:=C-\overline{A}$ and $\sigma(\overline{B},\mathcal{T}^2_{\mathcal{K}}(\overline{A},D))$ is the "sigma" term of $\mathcal{K}$, cf. e.g., **Bonnans and Shapiro** (2000).

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C^2-cone reducibility

Definition

The closed convex set $\mathcal{K}$ is said to be C^2-cone reducible at $\overline{A} \in \mathcal{K}$, if there exist a open neighborhood $\mathcal{W} \subset \mathcal{Y}$ of $\overline{A}$, a pointed closed convex cone $\mathcal{Q}$ (a cone is said to be pointed if and only if its lineality space is the origin) in a finite dimensional space $\mathcal{Z}$ and a twice continuously differentiable mapping $\Xi : \mathcal{W} \to \mathcal{Z}$ such that: (i) $\Xi(\overline{A}) = 0 \in \mathcal{Z}$; (ii) the derivative mapping $\Xi'(\overline{A}) : \mathcal{Y} \to \mathcal{Z}$ is onto; (iii) $\mathcal{K} \cap \mathcal{W} = \{A \in \mathcal{W} \mid \Xi(A) \in \mathcal{Q}\}$. We say that $\mathcal{K}$ is C^2-cone reducible if $\mathcal{K}$ is C^2-cone reducible at every $\overline{A} \in \mathcal{K}$.

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[14] Y. Cui, C. Ding and X.Y. Zhao, Quadratic Growth Conditions for Convex Matrix Optimization Problems Associated with Spectral Functions, to appear in SIAM Journal on Optimization (2017).

Main results

Robust isolated calm = isolated calm + lower semi-continuous

The lower semi-continuity of S_{KKT}

Proposition

Suppose that $\bar{x}$ is an isolated locally optimal solution with $(a, b) = (0, 0)$ and the corresponding set of Lagrange multipliers $M(\bar{x}, 0, 0) \neq \emptyset$. If the strict Robinson CQ holds at $\bar{x}$ with respect to $\bar{y} \in M(\bar{x}, 0, 0)$, then the KKT solution mapping S_{KKT} is lower semi-continuous at $(0, 0, \bar{x}, \bar{y}) \in \operatorname{gph} S_{\mathrm{KKT}}$.

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- The strict Robinson CQ (SRCQ) is said to hold with $(a, b) = (0, 0)$ at $\bar{x}$ with respect to $\bar{y} \in M(\bar{x}, 0, 0) \neq \emptyset$ if

$$G'(\bar{x})\mathcal{X} + \mathcal{T}_{\mathcal{K}}(G(\bar{x})) \cap \bar{y}^{\perp} = \mathcal{Y}.$$

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- The set of Lagrange multipliers $M(\bar{x}, 0, 0)$ is a singleton if the SRCQ holds.

The isolatedness of X_{KKT}

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Proposition

Suppose that the RCQ holds at a locally optimal solution $\bar{x}$ with $(a, b) = (0, 0)$ and that Robinson's SOSC

$$\inf_{y \in M(\bar{x},0,0)} \left\{ \langle d, \nabla^2_{xx} L(\bar{x}; y) d \rangle - \sigma\left(y, \mathcal{T}^2_{\mathcal{K}}(G(\bar{x}), G'(\bar{x})d) \right) \right\} > 0 \quad \forall\, d \in \mathcal{C}(\bar{x}) \backslash \{0\}$$

holds at $\bar{x}$. Then, there exists an open neighborhood $\mathcal{V}$ of $\bar{x}$ such that $X_{\mathrm{KKT}}(0,0) \cap \mathcal{V} = \{x\}$, which implies that $\bar{x}$ is an isolated locally optimal solution with $(a, b) = (0, 0)$.

The equivalent reformulation

When $(a, b) = (0, 0)$, the KKT system is equivalent to the following system of nonsmooth equations:

$$F(x, y) = 0,$$

where $F : \mathcal{X} \times \mathcal{Y} \to \mathcal{X} \times \mathcal{Y}$ is the natural mapping defined by

$$F(x, y) := \begin{bmatrix} \nabla f(x) + G'(x)^* y \\ G(x) - \Pi_{\mathcal{K}}(G(x) + y) \end{bmatrix}, \quad (x, y) \in \mathcal{X} \times \mathcal{Y}.$$

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Lemma

Let $(0, 0, \bar{x}, \bar{y}) \in \operatorname{gph} S_{\rm KKT}$. The set-valued mapping $S_{\rm KKT}$ is isolated calm at the origin for $(\bar{x}, \bar{y})$ if and only if the set-valued mapping F^{-1} is isolated calm at the origin for $(\bar{x}, \bar{y})$.

The characterization of the robust isolated calmness

Theorem

Let $\bar{x}$ be a feasible solution with $(a, b) = (0, 0)$. Suppose that the RCQ holds at $\bar{x}$. Assume that $\mathcal{K}$ is C^2-cone reducible at $G(\bar{x})$ with respect to $\bar{y} \in M(\bar{x}, 0, 0) \neq \emptyset$. Then the following statements are equivalent:

(i) *the SRCQ holds at $\bar{x}$ with respect to $\bar{y}$ and the SOSC holds at $\bar{x}$ with $(a, b) = (0, 0)$;*

(ii) *$\bar{x}$ is a locally optimal solution with $(a, b) = (0, 0)$ and $S_{\rm KKT}$ is robustly isolated calm at the origin for $(\bar{x}, \bar{y})$;*

(iii) *$\bar{x}$ is a locally optimal solution with $(a, b) = (0, 0)$ and $S_{\rm KKT}$ is isolated calm at the origin for $(\bar{x}, \bar{y})$.*

Error bound

The isolated calmness of the mapping F^{-1} at the origin for $(\bar{x}, \bar{y})$ implies the following error bound result: there exist a constant $\kappa > 0$ and a neighborhood $\mathcal{V}$ of $(\bar{x}, \bar{y})$ in $\mathcal{X} \times \mathcal{Y}$ such that

$$\|(x, y) - (\bar{x}, \bar{y})\| \leq \kappa \|F(x, y)\| \quad \forall\, (x, y) \in \mathcal{V}.$$

Robust isolated calmness v.s. Aubin property

By combining **Fusek** (2013), **Klatte and Kummer** (2013) and **Fusek** (2001), we obtain

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- *the constraint non-degeneracy condition holds at $\bar{x}$;*
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The constraint non-degeneracy is stronger than the SRCQ.

Another example

$$\begin{aligned} \min \quad & \tfrac{1}{2}(X_{11} - 1)^2 + \tfrac{1}{2}(X_{22} - 2X_{12})^2 \\ \text{s.t.} \quad & \langle E, X \rangle \leq 1, \\ & X \in \mathcal{S}_+^2. \end{aligned}$$

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- $\overline{X} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is the unique optimal solution and $(\bar{s}, \overline{Y}) = (0, 0) \in \Re \times \mathcal{S}^2$ is the unique corresponding Lagrange multiplier.

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- Both SRCQ and SOSC hold, which implies $S_{\rm KKT}$ is robustly isolated calm at the origin for $(\overline{X}, \bar{s}, \overline{Y})$.

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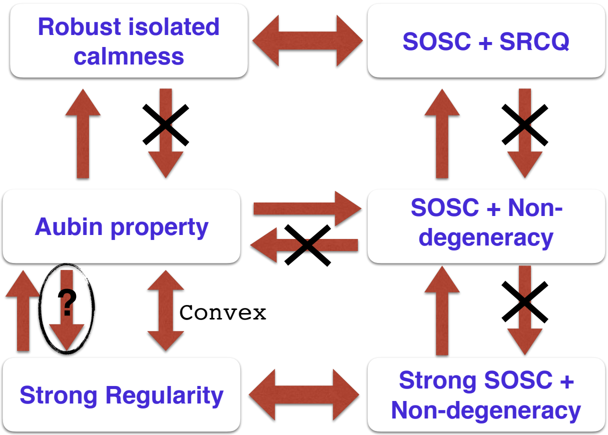
- Both SRCQ and SOSC hold, which implies $S_{\rm KKT}$ is robustly isolated calm at the origin for $(\overline{X}, \bar{s}, \overline{Y})$.
- Aubin property/strong regularity of $S_{\rm KKT}$ fails to hold since the strong SOSC does not hold.

Conclusions



Thank you