First Order Optimality Conditions for Mathematical Programs with SDP Cone Complementarity Constraints

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The SDCMPCC

The mathematical program with SDP cone complementarity constraints:

$$
\begin{array}{llll}
\text{(SDCMPCC)} & \min & f(z) & \\
& \text{s.t.} & h(z) = 0\,, & \\
& & g(z) \preceq_{\mathcal{Q}} 0\,, & \\
& & \mathcal{S}_+^{n_i} \ni G_i(z) \perp H_i(z) \in \mathcal{S}_-^{n_i}\,, & i = 1, \ldots, m\,,
\end{array}
$$

- $\mathcal{Z}$ and $\mathcal{H}$: two finite dimensional real Euclidean spaces
- $f : \mathcal{Z} \to \mathbb{R}$, $h : \mathcal{Z} \to \mathbb{R}^p$, $g : \mathcal{Z} \to \mathcal{H}$ and $G_i : \mathcal{Z} \to \mathcal{S}^{n_i}$, $H_i : \mathcal{Z} \to \mathcal{S}^{n_i}$ are continuously differentiable.
- $\mathcal{Q} \in \mathcal{H}$ is a closed convex symmetric cone with a nonempty interior, e.g.,
  - $\mathbb{R}_+^n$
  - the second-order cone $\mathcal{K}^n$
  - $\mathcal{S}_+^n$

Motivating examples

The rank constrained nearest correlation matrix problem:

$$\begin{array}{ll} \min & f(X) \\ \text{s.t.} & X_{ii} = 1, \quad i = 1, \ldots, n\,, \\ & X \in \mathcal{S}^n_+\,, \\ & \text{rank}(X) \leq r\,, \end{array}$$

where $f : \mathcal{S}^n \to \mathbb{R}$ is a given cost function that measures the closeness of X to a targeted matrix.

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The equivalent SDCMPCC reformulation:

$$\begin{array}{ll} \min\limits_{X,U} & f(X) \\ \text{s.t.} & X_{ii} = 1, \quad i = 1, \ldots, n\,, \\ & \langle I, U \rangle = r, \quad U \in \mathcal{S}^n_+\,, \\ & \mathcal{S}^n_+ \ni X \perp (U - I) \in \mathcal{S}^n_-\,. \end{array}$$

The bilinear matrix inequality (BMI) constraint optimization problem:

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & D + \textstyle\sum_{i=1}^{n} x_i \overline{A}^{(i)} + \sum_{i,j=1}^{n} W_{ij} \overline{C}^{(ij)} \preceq 0\,, \\ & W = x x^T\,. \end{aligned}$$

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Note that

$$W = xx^T \quad \Longleftrightarrow \quad Z = \begin{bmatrix} W & x \\ x^T & 1 \end{bmatrix} \succeq 0 \quad \text{and} \quad \text{rank}(Z) \leq 1\,.$$

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The similar problem so-called **phase retrieval problem** (cf. the talk given by E. Candès) also can be written as a SDCMPCC.

Robust bilevel programming, e.g., single-firm model in electric power market with uncertain data:

$$\begin{array}{ll} \min & f(\alpha, y) \\ \text{s.t.} & 0 \leq \alpha \leq \bar{\alpha}\,, \\ & \begin{array}{ll} \min & g(\alpha, y) \\ \text{s.t.} & \phi(\alpha, y, \zeta) \leq 0 \quad \forall\, \zeta \in \mathcal{U}\,, \\ & \mathcal{B}_1 y = 0, \quad 0 \leq \mathcal{B}_2 y \leq \bar{w}\,, \end{array} \end{array}$$

where $\mathcal{U}$ is some "uncertainty set", e.g., the ellipsoidal uncertainty set.

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Consider the robust counterpart of the lower-level problem and the corresponding KKT conditions, we obtain the following SDCMPCC:

$$\begin{array}{ll} \min & f(\alpha, y) \\ \text{s.t.} & 0 \le \alpha \le \bar{\alpha}\,, \\ & \psi(y, \xi, \eta, \Gamma) = 0, \quad \mathcal{B}_1 y = 0\,, \\ & 0 \le \eta \perp -\mathcal{B}_2 y \le 0, \quad 0 \le \zeta \perp \mathcal{B}_2 y - \bar{w} \le 0\,, \\ & 0 \preceq \mathcal{A}_\alpha y \perp \Gamma \preceq 0\,. \end{array}$$

The vector MPCC can be consider as a special case of SDCMPCC.

$$\mathcal{S}^1_+ \ni G_i(z) \perp H_i(z) \in \mathcal{S}^1_-, \quad i = 1, \ldots, n\,.$$

The vector MPCC:

- MFCQ fails
- Strong stationary conditions (S-stationary conditions).
- Mordukhovich stationary conditions (M-stationary conditions).
- Clarke stationary conditions (C-stationary conditions).

What about the SDCMPCC?

Outline

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Conic programming approach

The complementarity constraint in SDCMPCC is equivalent to the following

$$\langle G(z), H(z)\rangle \geq 0, \quad G(z) \in \mathcal{S}^n_+, \quad H(z) \in \mathcal{S}^n_- \,.$$

For any $G(z) \in \mathcal{S}^n_+$ and $H(z) \in \mathcal{S}^n_-$, by Fan's inequality, we have

$$\langle G(z), H(z)\rangle \leq \lambda(G(z))^T \lambda(H(z)) \leq 0 \,.$$

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$$\text{(CP-SDCMPCC)} \qquad \begin{array}{ll} \min & f(z) \\ \text{s.t.} & h(z) = 0 \,, \\ & g(z) \preceq_{\mathcal{Q}} 0 \,, \\ & \langle G(z), H(z)\rangle \geq 0 \,, \\ & G(z) \in \mathcal{S}^n_+, \; H(z) \in \mathcal{S}^n_- \,. \end{array}$$

The classical KKT condition

Definition 1

Let $\bar{z}$ be a feasible solution of SDCMPCC. We call $\bar{z}$ a classical KKT point. If there exists $(\lambda^h, \lambda^g, \lambda^e, \Omega^G, \Omega^H) \in \mathbb{R}^p \times \mathcal{H} \times \mathbb{R} \times \mathcal{S}^n \times \mathcal{S}^n$ with $\lambda^g \in \mathcal{Q}$, $\lambda^e \leq 0$, $\Omega^G \preceq 0$ and $\Omega^H \succeq 0$ such that

$$\begin{aligned}
0 = \nabla f(\bar{z}) + h'(\bar{z})^* \lambda^h + g'(\bar{z})^* \lambda^g + \lambda^e [H'(\bar{z})^* G(\bar{z}) + G'(\bar{z})^* H(\bar{z})] \\
+ G'(\bar{z})^* \Omega^G + H'(\bar{z})^* \Omega^H, \\
\langle g(\bar{z}), \lambda^g \rangle = 0, \quad G(\bar{z}) \Omega^G = 0, \quad H(\bar{z}) \Omega^H = 0\,.
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Definition 1

Let $\bar{z}$ be a feasible solution of SDCMPCC. We call $\bar{z}$ a classical KKT point. If there exists $(\lambda^h, \lambda^g, \lambda^e, \Omega^G, \Omega^H) \in \mathbb{R}^p \times \mathcal{H} \times \mathbb{R} \times \mathcal{S}^n \times \mathcal{S}^n$ with $\lambda^g \in \mathcal{Q}$, $\lambda^e \leq 0$, $\Omega^G \preceq 0$ and $\Omega^H \succeq 0$ such that

$$\begin{aligned} 0 = & \nabla f(\bar{z}) + h'(\bar{z})^* \lambda^h + g'(\bar{z})^* \lambda^g + \lambda^e [H'(\bar{z})^* G(\bar{z}) + G'(\bar{z})^* H(\bar{z})] \\ & + G'(\bar{z})^* \Omega^G + H'(\bar{z})^* \Omega^H, \\ & \langle g(\bar{z}), \lambda^g \rangle = 0, \quad G(\bar{z}) \Omega^G = 0, \quad H(\bar{z}) \Omega^H = 0\,. \end{aligned}$$

Proposition 1 (Failure of Robinson's CQ)

For CP-SDCMPCC, Robinson's constraint qualification fails to hold at every feasible solution of SDCMPCC.

Theorem 1

Let $\bar{z}$ be a local optimal solution of SDCMPCC. Suppose that the problem CP-SDCMPCC is **Clarke calm** *at $\bar{z}$; in particular the set-valued map*

$$\begin{aligned}\mathcal{F}(r,s,t,P) \quad := \quad & \{z \mid h(z)+r=0, \quad g(z)+s \preceq_{\mathcal{Q}} 0, \\ & -\langle G(z),H(z)\rangle + t \le 0, (G(z),H(z))+P \in \mathcal{S}^n_+ \times \mathcal{S}^n_-\}\end{aligned} \tag{1}$$

is **calm** *at $(0,0,0,0,\bar{z})$. Then $\bar{z}$ is a classical KKT point.*

Clarke calm at a local optimal solution $\bar{z}$: $\exists\, \varepsilon > 0, \mu > 0$ such that, for all (r,s,t,P) in εB, for all $z \in (\bar{z}+\varepsilon B) \cap \mathcal{F}(r,s,t,P)$, one has

$$f(z) - f(\bar{z}) + \mu\|(r,s,t,P)\| \ge 0\,.$$

Calmness of a set-valued map:

$\Phi : \mathcal{Z} \rightrightarrows \mathcal{Y}$ is said to be calm at a point $(\bar{z},\bar{v}) \in \operatorname{gph}\Phi$, if $\exists$ a constant $M > 0$ and two neighborhoods U of $\bar{z}$ and V of $\bar{v}$ such that

$$\Phi(z) \cap V \subseteq \Phi(\bar{z}) + M\|z-\bar{z}\| \operatorname{cl} B \quad \forall\, z \in U.$$

Outline

The general optimization approach

$$
\text{(GP-SDCMPCC)} \quad \begin{array}{ll} \min & f(z) \\ \text{s.t.} & h(z) = 0\,, \\ & g(z) \preceq_{\mathcal{Q}} 0\,, \\ & (G(z), H(z)) \in \operatorname{gph} N_{\mathcal{S}^n_+}\,, \end{array}
$$

where

$$
\operatorname{gph} N_{\mathcal{S}^n_+} = \{(X, Y) \in \mathcal{S}^n_+ \times \mathcal{S}^n_- : \Pi_{\mathcal{S}^n_+}(X + Y) = X,\ \Pi_{\mathcal{S}^n_-}(X + Y) = Y\}\,.
$$

The general optimization approach

$$\text{(GP-SDCMPCC)} \quad \begin{array}{ll} \min & f(z) \\ \text{s.t.} & h(z) = 0\,, \\ & g(z) \preceq_{\mathcal{Q}} 0\,, \\ & (G(z), H(z)) \in \operatorname{gph} N_{\mathcal{S}^n_+}\,, \end{array}$$

where

$$\operatorname{gph} N_{\mathcal{S}^n_+} = \{(X,Y) \in \mathcal{S}^n_+ \times \mathcal{S}^n_- : \Pi_{\mathcal{S}^n_+}(X+Y) = X,\ \Pi_{\mathcal{S}^n_-}(X+Y) = Y\}\,.$$

—— $N_{\mathcal{S}^n_+}(\cdot)$ is the **normal cone mapping** of the closed convex cone $\mathcal{S}^n_+$, i.e., if $X \in \mathcal{S}^n_+$,

$$N_{\mathcal{S}^n_+}(X) = \left\{Y \in \mathcal{S}^n : \langle Y, Z - X\rangle \leq 0 \quad \forall\, Z \in \mathcal{S}^n_+\right\}\,,$$

otherwise, $N_{\mathcal{S}^n_+}(X) \equiv \emptyset$.

The key step: derive the explicit expression of the limiting normal cone $N_{\operatorname{gph} N_{\mathcal{S}^n_+}}(G(\bar{z}), H(\bar{z}))$

Normal cones

X: a finite dimensional Euclidean space

Ω: a nonempty subset of $\mathcal{X}$

Let $\bar{x} \in \mathrm{cl}\,\Omega$ be given.

- **Proximal normal cone**:

$$N_\Omega^\pi(\bar{x}) := \left\{\zeta \in \mathcal{X} \mid \exists\, M > 0, \text{ s.t. } \langle \zeta, x - \bar{x} \rangle \le M\|x - \bar{x}\|^2 \ \forall\, x \in \Omega\right\} .$$

- **Limiting normal cone (Mordukhovich normal cone)**:

$$N_\Omega(\bar{x}) := \{\lim_{i\to\infty} \zeta_i : \zeta_i \in N_\Omega^\pi(x_i), \ \ x_i \to \bar{x}, \ \ x_i \in \Omega\} .$$

For any given $(X, Y) \in \operatorname{gph} N_{\mathcal{S}^n_+}$, let $A := X + Y$ have the eigenvalue decomposition

$$A = \overline{P}\Lambda(A)\overline{P}^T \,. \tag{2}$$

Define the three index sets

$$\alpha := \{i : \lambda_i(A) > 0\}, \quad \beta := \{i : \lambda_i(A) = 0\} \quad \text{and} \quad \gamma := \{i : \lambda_i(A) < 0\}\,.$$

Proposition 2

The metric projection operator $\Pi_{\mathcal{S}^n_+}(\cdot)$ is 1-order B-differentiable, for any given $A \in \mathcal{S}^n$, i.e., for $\mathcal{S}^n \ni H \to 0$,

$$\Pi_{\mathcal{S}^n_+}(A + H) - \Pi_{\mathcal{S}^n_+}(A) - \Pi'_{\mathcal{S}^n_+}(A; H) = O(\|H\|^2)\,,$$

where $\Pi'_{\mathcal{S}^n_+}(A; \cdot)$ is the directional derivative of $\Pi_{\mathcal{S}^n_+}(\cdot)$ at A.

The proximal normal cone

Proposition 3

$$N^{\pi}_{\mathrm{gph}\, N_{\mathcal{S}^n_+}}(X,Y) = \Big\{(X^*,Y^*) \in \mathcal{S}^n \times \mathcal{S}^n : \Theta_1 \circ \widetilde{X}^* + \Theta_2 \circ \widetilde{Y}^* = 0, \quad \widetilde{X}^*_{\beta\beta} \preceq 0 \quad \text{and} \quad \widetilde{Y}^*_{\beta\beta} \succeq 0\Big\},$$

where $\widetilde{X}^* := \overline{P}^T X^* \overline{P}$ *and* $\widetilde{Y}^* := \overline{P}^T Y^* \overline{P}$.

$$\Theta_1 := \begin{bmatrix} E_{\alpha\alpha} & E_{\alpha\beta} & \Sigma_{\alpha\gamma} \\ E_{\alpha\beta}^T & 0 & 0 \\ \Sigma_{\alpha\gamma}^T & 0 & 0 \end{bmatrix} \quad \text{and} \quad \Theta_2 := \begin{bmatrix} 0 & 0 & E_{\alpha\gamma} - \Sigma_{\alpha\gamma} \\ 0 & 0 & E_{\beta\gamma} \\ (E_{\alpha\gamma} - \Sigma_{\alpha\gamma})^T & E_{\beta\gamma}^T & E_{\gamma\gamma} \end{bmatrix}.$$

where $\Sigma_{ij} := \frac{\max\{\lambda_i(A),0\} - \max\{\lambda_j(A),0\}}{\lambda_i(A) - \lambda_j(A)}$.

The limiting normal cone

Firstly, we will characterize $N_{\mathrm{gph}\, N_{\mathcal{S}_+^{|\beta|}}}(0,0)$ for the case that $\beta \neq \emptyset$.

Proposition 4

The limiting norm cone to the graph of the normal cone mapping $N_{\mathcal{S}_+^{|\beta|}}$ *at* $(0,0)$ *is given by*

$$N_{\mathrm{gph}\, N_{\mathcal{S}_+^{|\beta|}}}(0,0) = \bigcup_{\substack{Q \,\in\, \mathcal{O}^{|\beta|} \\ \Xi_1,\, \Xi_2 \,\in\, \mathcal{U}_{|\beta|}}} \left\{ \begin{array}{ll} (U^*, V^*): & \Xi_1 \circ Q^T U^* Q + \Xi_2 \circ Q^T V^* Q = 0, \\ & Q_{\beta_0}^T U^* Q_{\beta_0} \preceq 0, \quad Q_{\beta_0}^T V^* Q_{\beta_0} \succeq 0 \end{array} \right\},$$

If $\Xi_1, \Xi_2 \in \mathcal{U}_{|\beta|}$, then there exists a partition $\pi(\beta) := (\beta_+,\, \beta_0,\, \beta_-) \in \mathscr{P}(\beta)$ such that

$$\Xi_1 = \left[\begin{array}{ccc} E_{\beta_+\beta_+} & E_{\beta_+\beta_0} & (\Xi_1)_{\beta_+\beta_-} \\ E^T_{\beta_+\beta_0} & 0 & 0 \\ (\Xi_1)^T_{\beta_+\beta_-} & 0 & 0 \end{array} \right],$$

where each element of $(\Xi_1)_{\beta_+\beta_-}$ belongs to $[0, 1]$, and

$$\Xi_2 := \left[\begin{array}{ccc} 0 & 0 & E_{\beta_+\beta_-} - (\Xi_1)_{\beta_+\beta_-} \\ 0 & 0 & E_{\beta_0\beta_-} \\ (E_{\beta_+\beta_-} - (\Xi_1)_{\beta_+\beta_-})^T & E^T_{\beta_0\beta_-} & E_{\beta_-\beta_-} \end{array} \right].$$

Theorem 2

For any $(X,Y) \in \operatorname{gph} N_{\mathcal{S}^n_+}$, *let* $A = X + Y$ *have the eigenvalue decomposition (2). Then,* $(X^*,Y^*) \in N_{\operatorname{gph} N_{\mathcal{S}^n_+}}(X,Y)$ *if and only if*

$$X^* = \overline{P}\begin{bmatrix} 0 & 0 & \widetilde{X}^*_{\alpha\gamma} \\ 0 & \widetilde{X}^*_{\beta\beta} & \widetilde{X}^*_{\beta\gamma} \\ \widetilde{X}^*_{\gamma\alpha} & \widetilde{X}^*_{\gamma\beta} & \widetilde{X}^*_{\gamma\gamma} \end{bmatrix}\overline{P}^T \quad \text{and} \quad Y^* = \overline{P}\begin{bmatrix} \widetilde{Y}^*_{\alpha\alpha} & \widetilde{Y}^*_{\alpha\beta} & \widetilde{Y}^*_{\alpha\gamma} \\ \widetilde{Y}^*_{\beta\alpha} & \widetilde{Y}^*_{\beta\beta} & 0 \\ \widetilde{Y}^*_{\gamma\alpha} & 0 & 0 \end{bmatrix}\overline{P}^T$$

with

$$(\widetilde{X}^*_{\beta\beta}, \widetilde{Y}^*_{\beta\beta}) \in N_{\operatorname{gph} N_{\mathcal{S}^{|\beta|}_+}}(0,0) \quad \text{and} \quad \Sigma_{\alpha\gamma} \circ \widetilde{X}^*_{\alpha\gamma} + (E_{\alpha\gamma} - \Sigma_{\alpha\gamma}) \circ \widetilde{Y}^*_{\alpha\gamma} = 0\,,$$

$\widetilde{X}^* = \overline{P}^T X^* \overline{P}$ *and* $\widetilde{Y}^* = \overline{P}^T Y^* \overline{P}$.

Coderivatives

- For any given $(X, Y) \in \operatorname{gph} N_{\mathcal{S}^n_+}$, the (Mordukhovich) coderivative $D^* N_{\mathcal{S}^n_+}(X, Y)$ of the normal cone to the set $\mathcal{S}^n_+$ can be characterised as follows

$$X^* \in D^* N_{\mathcal{S}^n_+}(X, Y)(Y^*) \quad \Longleftrightarrow \quad (X^*, -Y^*) \in N_{\operatorname{gph} N_{\mathcal{S}^n_+}}(X, Y)\,.$$

- For any given $U^* \in \mathcal{S}^n$, $V^* \in D^* \Pi_{\mathcal{S}^n_+}(X + Y)(U^*)$ if and only if there exists $(X^*, Y^*) \in N_{\operatorname{gph} N_{\mathcal{S}^n_+}}(X, Y)$ such that

$$X^* = V^* - U^* \quad \text{and} \quad Y^* = V^*\,.$$

—— The explicit expression of $D^* \Pi_{\mathcal{S}^n_+}(X + Y)(U^*)$ is provided.

The S-stationary condition

For the vector MPCC, the S-stationary condition is shown to be equivalent to the necessary optimality condition of a reformulated problem involving the proximal normal cone to the graph of the normal cone operator (see Theorem 3.2 [Ye, 1999]).

Definition 2

Let $\bar{z}$ be a feasible solution of SDCMPCC. Let $A := G(\bar{z}) + H(\bar{z})$ have the eigenvalue decomposition (2). We say that $\bar{z}$ is a S-stationary point of SDCMPCC if there exists $(\lambda^h, \lambda^g, \Gamma^G, \Gamma^H) \in \mathbb{R}^p \times \mathcal{H} \times \mathcal{S}^n \times \mathcal{S}^n$ such that

$$0 = \nabla f(\bar{z}) + h'(\bar{z})^* \lambda^h + g'(\bar{z})^* \lambda^g + G'(\bar{z})^* \Gamma^G + H'(\bar{z})^* \Gamma^H \,, \tag{3}$$

$$\lambda^g \in \mathcal{Q}, \quad \langle \lambda^g, g(\bar{z}) \rangle = 0 \,, \tag{4}$$

$$\widetilde{\Gamma}^G_{\alpha\alpha} = 0, \quad \widetilde{\Gamma}^G_{\alpha\beta} = 0, \quad \widetilde{\Gamma}^G_{\beta\alpha} = 0 \,, \tag{5}$$

$$\widetilde{\Gamma}^H_{\gamma\gamma} = 0, \quad \widetilde{\Gamma}^H_{\beta\gamma} = 0, \quad \widetilde{\Gamma}^H_{\gamma\beta} = 0 \,, \tag{6}$$

$$\Sigma_{\alpha\gamma} \circ \widetilde{\Gamma}^G_{\alpha\gamma} + (E_{\alpha\gamma} - \Sigma_{\alpha\gamma}) \circ \widetilde{\Gamma}^H_{\alpha\gamma} = 0 \,, \tag{7}$$

and

$$\widetilde{\Gamma}^G_{\beta\beta} \preceq 0, \quad \widetilde{\Gamma}^H_{\beta\beta} \succeq 0 \,, \tag{8}$$

where $\widetilde{\Gamma}^G = \overline{P}^T \Gamma^G \overline{P}$ and $\widetilde{\Gamma}^H = \overline{P}^T \Gamma^H \overline{P}$.

Proposition 5

Let $\bar{z}$ be a feasible solution of SDCMPCC. If $\bar{z}$ is a classic KKT point, then it is also a S-stationary point.

Corollary 1

Let $\bar{z}$ be an optimal solution of SDCMPCC. Suppose the problem CP-SDCMPCC is Clarke calm at $\bar{z}$; in particular the set-valued map defined by (1) is calm at $(0, 0, 0, 0, \bar{z})$. Then $\bar{z}$ is a S-stationary point.

The M-stationary condition

Definition 3

Let $\bar{z}$ be a feasible solution of SDCMPCC. Let $A = G(\bar{z}) + H(\bar{z})$ have the eigenvalue decomposition (2). We say that $\bar{z}$ is a M-stationary point of SDCMPCC if there exists $(\lambda^h, \lambda^g, \Gamma^G, \Gamma^H) \in \mathbb{R}^p \times \mathcal{H} \times \mathcal{S}^n \times \mathcal{S}^n$ such that (3)-(7) hold and there exist $Q \in \mathcal{O}^{|\beta|}$ and $\Xi_1, \Xi_2 \in \mathcal{U}_{|\beta|}$ such that (with a partition $\pi(\beta) = (\beta_+, \beta_0, \beta_-)$ of β)

$$\Xi_1 \circ Q^T \widetilde{\Gamma}^G Q + \Xi_2 \circ Q^T \widetilde{\Gamma}^H Q = 0 \,, \tag{9}$$

$$Q_{\beta_0}^T \widetilde{\Gamma}_{\beta\beta}^G Q_{\beta_0} \preceq 0, \quad Q_{\beta_0}^T \widetilde{\Gamma}_{\beta\beta}^H Q_{\beta_0} \succeq 0 \,, \tag{10}$$

The M-stationary condition (cont'd)

We say that $(\lambda^h, \lambda^g, \Gamma^G, \Gamma^H) \in \mathbb{R}^p \times \mathbb{R}^q \times \mathcal{S}^n \times \mathcal{S}^n$ is a singular M-multiplier for SDCMPCC if it is not equal to zero and all conditions above hold except the term $\nabla f(\bar{z})$ vanishes in (3).

Theorem 3

Let $\bar{z}$ be a local optimal solution of SDCMPCC. Suppose that either the problem GP-SDCMPCC is Clarke calm at $\bar{z}$ or one of the following constraint qualifications holds. Then $\bar{z}$ is a M-stationary point of SDCMPCC.

(i) *There is no singular M-multiplier for problem SDCMPCC at $\bar{z}$.*

(ii) *SDCMPCC LICQ holds at $\bar{z}$: there is no nonzero $(\lambda^h, \lambda^g, \Gamma^G, \Gamma^H) \in \mathbb{R}^p \times \mathcal{H} \times \mathcal{S}^n \times \mathcal{S}^n$ such that*

$$\begin{aligned}
&h'(\bar{z})^*\lambda^h + g'(\bar{z})^*\lambda^g + G'(\bar{z})^*\Gamma^G + H'(\bar{z})^*\Gamma^H = 0\,,\\
&\widetilde{\Gamma}^G_{\alpha\alpha} = 0, \quad \widetilde{\Gamma}^G_{\alpha\beta} = 0, \quad \widetilde{\Gamma}^G_{\beta\alpha} = 0\,,\\
&\widetilde{\Gamma}^H_{\gamma\gamma} = 0, \quad \widetilde{\Gamma}^H_{\beta\gamma} = 0, \quad \widetilde{\Gamma}^H_{\gamma\beta} = 0\,,\\
&\Sigma_{\alpha\gamma} \circ \widetilde{\Gamma}^G_{\alpha\gamma} + (E_{\alpha\gamma} - \Sigma_{\alpha\gamma}) \circ \widetilde{\Gamma}^H_{\alpha\gamma} = 0\,.
\end{aligned}$$

(iii) Assume that there is no inequality constraint $g(z) \preceq_{\mathcal{Q}} 0$. Assume also that $\mathcal{Z} = \mathcal{X} \times \mathcal{S}^n$ where X is a finite dimensional space and $G(x,u) = u$. The following generalized equation is strongly regular in the sense of Robinson:

$$0 \in -F(x,u) + N_{\mathbb{R}^q \times \mathcal{S}^n_+}(x,u)\,,$$

where $F(x,u) = (h(x,u), H(x,u))$.

(iv) Assume that there is no inequality constraint $g(z) \preceq_{\mathcal{Q}} 0$. Assume also that $\mathcal{Z} = \mathcal{X} \times \mathcal{S}^n$, $G(z) = u$ and $F(x,u) = (h(x,u), H(x,u))$. $-F$ is locally strongly monotone in u uniformly in x with modulus $\delta > 0$, i.e., there exist neighborhood U_1 of $\bar{x}$ and U_2 of $\bar{u}$ such that

$$\langle -F(x,u) + F(x,v), u - v \rangle \geq \delta \|u - v\|^2 \quad \forall\, u \in U_2 \cap \mathcal{S}^n_+,\ v \in \mathcal{S}^n_+,\ x \in U_1\,.$$

An example

Consider the following SDCMPCC problem

$$\begin{array}{ll} \min & -\langle I, X\rangle + \langle I, Y\rangle \\ \text{s.t.} & X + Y = 0\,, \\ & \mathcal{S}^n_+ \ni X \perp Y \in \mathcal{S}^n_-\,. \end{array}$$

Since the unique feasible point is $(0,0)$, we know that $(X^*, Y^*) = (0,0)$ is the optimal solution.

$$\alpha = \emptyset, \quad \beta = \{1, \ldots, n\} \quad \text{and} \quad \gamma = \emptyset\,.$$

WLOG, let $\overline{P} = I$.

$$\begin{bmatrix} -I \\ I \end{bmatrix} + \begin{bmatrix} \Gamma^e \\ \Gamma^e \end{bmatrix} + \begin{bmatrix} \Gamma^G \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \Gamma^H \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

- The optimal solution $(0,0)$ is a M-stationary point with the multiplier $\Gamma^e = I$, $\Gamma^G = 0$ and $\Gamma^H = -2I$ (with $\beta_+ = \beta = \{1, \ldots, n\}$, $\beta_0 = \beta_- = \emptyset$, and $Q = I \in \mathcal{O}^n$).
- $(0,0)$ is not a S-stationary point.

Outline

$$\text{(NS-SDCMPCC)} \qquad \begin{aligned} \min \quad & f(z) \\ \text{s.t.} \quad & h(z) = 0\,, \\ & g(z) \preceq_{\mathcal{Q}} 0\,, \\ & G(z) - \Pi_{\mathcal{S}^n_+}(G(z) + H(z)) = 0\,. \end{aligned}$$

Note that

$$\mathcal{S}^n_+ \ni G(z) \perp H(z) \in \mathcal{S}^n_- \quad \Longleftrightarrow \quad G(z) - \Pi_{\mathcal{S}^n_+}(G(z) + H(z)) = 0\,.$$

The C-stationary condition

Definition 4

Let $\bar{z}$ be a feasible solution of SDCMPCC. Let $A = G(\bar{z}) + H(\bar{z})$ have the eigenvalue decomposition (2). We say that $\bar{z}$ is a C-stationary point of SDCMPCC if there exists $(\lambda^h, \lambda^g, \Gamma^G, \Gamma^H) \in \mathbb{R}^p \times \mathcal{H} \times \mathcal{S}^n \times \mathcal{S}^n$ such that (3)-(7) hold and

$$\langle \widetilde{\Gamma}^G_{\beta\beta} , \widetilde{\Gamma}^H_{\beta\beta} \rangle \leq 0 \, ,$$

where $\widetilde{\Gamma}^G = \overline{P}^T \Gamma^G \overline{P}$ and $\widetilde{\Gamma}^H = \overline{P}^T \Gamma^H \overline{P}$. We say that $(\lambda^h, \lambda^g, \Gamma^G, \Gamma^H) \in \mathbb{R}^p \times \mathcal{H} \times \mathcal{S}^n \times \mathcal{S}^n$ is a singular C-multiplier for SDCMPCC if it is not equal to zero and all conditions above hold except the term $\nabla f(\bar{z})$ vanishes in (3).

Theorem 4

Let $\bar{z}$ be a local optimal solution of SDCMPCC. Suppose that the problem NS-SDCMPCC is Clarke calm at $\bar{z}$ or there is no singular C-multiplier for problem SDCMPCC at $\bar{z}$. Then $\bar{z}$ is a C-stationary point of SDCMPCC.

As in the vector MPCC case, we have

S-stationary condition $\Longrightarrow$ M-stationary condition $\Longrightarrow$ C-stationary condition

Another example

Consider the following SDCMPCC problem

$$
\begin{aligned}
\min \quad & \tfrac{1}{2}z_1 - \tfrac{1}{2}z_2 - z_3 - \tfrac{1}{2}z_4 \\
\text{s.t.} \quad & -2z_1 + z_3 + z_4 \le 0\,, \\
& 2z_2 + z_3 \le 0\,, \\
& z_4^2 \le 0\,, \\
& \mathcal{S}^3_+ \ni G(z) \perp H(z) \in \mathcal{S}^3_-\,,
\end{aligned}
$$

where

$$
G(z) := \begin{bmatrix} 1+\frac{z_1}{6} & -1+\frac{z_1}{6} & -\frac{z_1}{3} \\ -1+\frac{z_1}{6} & 1+\frac{z_1}{6} & -\frac{z_1}{3} \\ -\frac{z_1}{3} & -\frac{z_1}{3} & \frac{2}{3}z_1 \end{bmatrix} \quad \text{and} \quad H(z) := \begin{bmatrix} \frac{z_2}{6}-1 & \frac{z_2}{6}-1 & -\frac{z_2}{3}-1 \\ \frac{z_2}{6}-1 & \frac{z_2}{6}-1 & -\frac{z_2}{3}-1 \\ -\frac{z_2}{3}-1 & -\frac{z_2}{3}-1 & \frac{2}{3}z_2-1 \end{bmatrix}.
$$

Since $\langle G(z), H(z)\rangle = z_1 z_2$, one can verify that $\bar{z} = (0,0,0,0)$ is the unique optimal solution.

The index sets of positive, zero and negative eigenvalues:

$$\alpha = \{1\}, \quad \beta = \{2\} \quad \text{and} \quad \gamma = \{3\}$$

The equation (3) can be written as

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -1 \\ -\frac{1}{2} \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix} \lambda_1^g + \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} \lambda_2^g + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \lambda_3^g + \begin{bmatrix} \langle \frac{\partial G}{\partial z_1}, \Gamma^G \rangle \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \langle \frac{\partial H}{\partial z_2}, \Gamma^H \rangle \\ 0 \\ 0 \end{bmatrix}.$$

Therefore, we know the optimal solution $\bar{z} = (0,0,0,0)$ is a C-stationary point with the multiplier

$$\lambda^g = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix}, \quad \Gamma^G = \frac{1}{6} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -1 \\ \frac{1}{2} & \frac{1}{2} & -1 \\ -1 & -1 & 2 \end{bmatrix} \quad \text{and} \quad \Gamma^H = -\frac{1}{6} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -1 \\ \frac{1}{2} & \frac{1}{2} & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

However, we can prove that the optimal solution $\bar{z} = (0,0,0,0)$ is not a M-stationary point (since $\Gamma^G_{22} = \frac{1}{2} > 0$, $\Gamma^H_{22} = -\frac{1}{2} < 0$).

Outline

Conclusions

- The M-, S- and C-stationary conditions for SDCMPCC are characterised.
- Does S-stationary condition imply the classical KKT condition?
- Numerical algorithms
- Mordukhovich coderivative criterion and perturbation analysis of the nonlinear SDP problem

Thank you

To see that the S-stationary condition for SDCMPCC coincides with the S-stationary condition in the MPCC case, we consider the case when $n = 1$. In this case $\lambda(A) = A$, $\overline{P} = 1$ and $\widetilde{\Gamma}^G = \Gamma^G$, $\widetilde{\Gamma}^H = \Gamma^H$.

- If $G(\bar{z}) > 0, H(\bar{z}) = 0$, then $\beta = \gamma = \emptyset$ and from (5), we know that $\Gamma^G = 0$ and Γ^H free.
- If $G(\bar{z}) = 0, H(\bar{z}) < 0$, then $\Gamma^H = 0$ and Γ^G free, similarly.
- If $G(\bar{z}) = H(\bar{z}) = 0$, then $\alpha = \gamma = \emptyset$ and from (8), we know that $\Gamma^G \leq 0$ and $\Gamma^H \geq 0$.

To see that the M-stationary condition for SDCMPCC coincides with the M-stationary condition in the MPCC case, we consider the case when $n = 1$. In this case $\lambda(A) = A$, $\overline{P} = 1$ and $\widetilde{\Gamma}^G = \Gamma^G$, $\widetilde{\Gamma}^H = \Gamma^H$ and SDCMPCC is a MPCC where there is only one complementarity constraint.

- If $G(\bar{z}) > 0, H(\bar{z}) = 0$, then $\beta = \gamma = \emptyset$ and from (5), we have $\Gamma^G = 0$ and Γ^H free.
- If $G(\bar{z}) = 0, H(\bar{z}) < 0$, then $\Gamma^H = 0$ and Γ^G free, similarly.
- If $G(\bar{z}) = H(\bar{z}) = 0$, then $\alpha = \gamma = \emptyset$. Let $\pi(\beta) = (\beta_+, \beta_0, \beta_-)$ be a partition of β. We know that there are only three cases:
  - **Case 1:** $\beta = \beta_+ \neq \emptyset$. From (9), $\Gamma^G = 0$.
  - **Case 2:** $\beta = \beta_- \neq \emptyset$. From (9), $\Gamma^H = 0$.
  - **Case 3:** $\beta = \beta_0 \neq \emptyset$. From (10), $\Gamma^G \leq 0$ and $\Gamma^H \geq 0$.

  Therefore, in any case, we have

$$\text{either } \Gamma^G < 0,\ \Gamma^H > 0 \quad \text{or} \quad \Gamma^G \Gamma^H = 0.$$