On the convergence analysis of augmented Lagrangian method for matrix optimization

Strong variational sufficiency for nonlinear SDP

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Optimization in the Big Data Era
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S.W. Wang, **D.**, Y.J. Zhang, and X.Y. Zhao, *Strong variational sufficiency for nonlinear semidefinite programming and its implications*, arXiv:2210.04448, 2022.

Outline

Generalized convexity

(Strong) variational sufficiency

Characterization of strong variational sufficiency I

ALM for NLSDP

Characterization of strong variational sufficiency II

Numerical experiments

Generalized convexity

Convexity

Definition

Let $f : \mathbb{X} \to [-\infty, +\infty]$. We define f to be a **convex** function on $\mathrm{dom} f$ if $\mathrm{epi} f$ is convex.

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$$\langle v' - v, x' - x \rangle \geq 0 \text{ for all } (x, v), (x', v') \in \mathrm{gph} T.$$

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$$\boxed{f \text{ is convex}} \iff \boxed{\partial f \text{ is (maximal) monotone}}$$

$\hat{\partial} f$: $f(x') \geq f(x) + \langle v, x' - x \rangle + o(\|x' - x\|)$
∂f: $\exists v^k \in \hat{\partial} f(x^k)$ such that $v^k \to v$, $x^k \to x$, $f(x^k) \to f(x)$

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Nonconvexity may lose the above properties

Beyond convexity (I): local convexity

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If f is convex relative to some neighborhood $\mathcal{X}$ of a given $\bar{x}$, then f is a **locally convex** function.

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A mapping T is **locally monotone** if there exists a neighborhood $\mathcal{X} \times \mathcal{V}$ of given $(\bar{x}, \bar{v})$ such that

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Beyond convexity (I): local convexity (con't)

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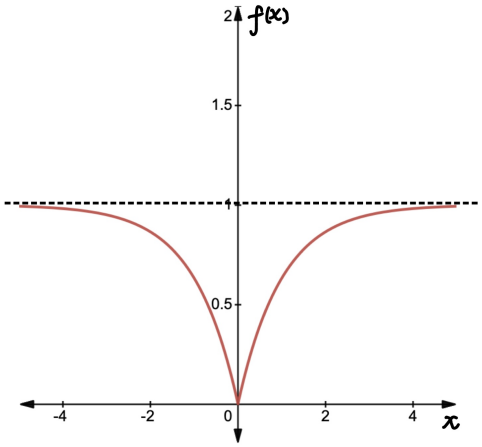
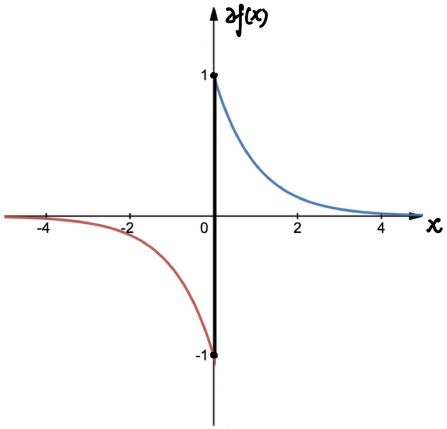
Example 1: $f(x) = \max\{1 - e^x, 1 - e^{-x}\}$, $\partial f(x) = \begin{cases} \{-e^x\}, & x < 0 \\ [-1, 1], & x = 0 \\ \{e^{-x}\}, & x > 0 \end{cases}$



(a)



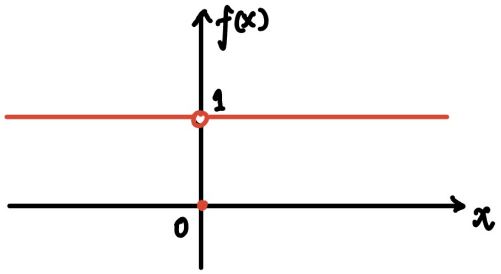
(b)

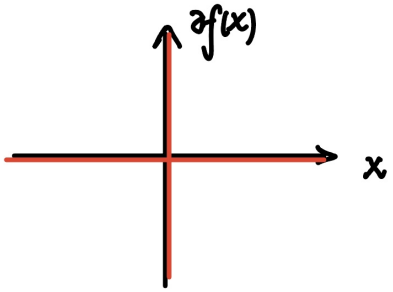
- f is **concave** on $(-\infty, 0]$ and $[0, \infty)$
- $0 \in \partial f(0)$; on a neighborhood of $(0, 0)$, ∂f is **locally maximal monotone**

Beyond convexity (II): variational convexity

Example 2:

$$f(x) = \begin{cases} 0, & x = 0 \\ 1, & x \neq 0 \end{cases}$$





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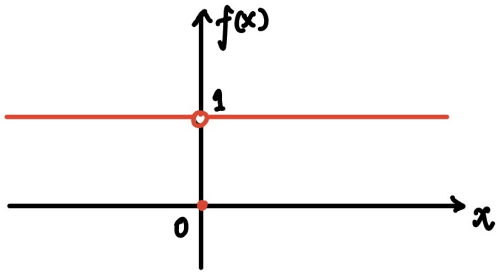

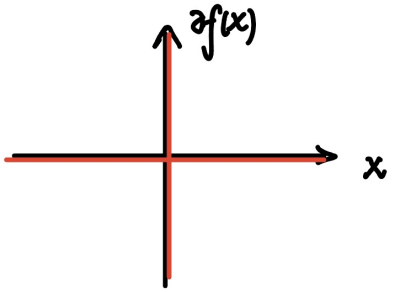

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An observation: alternating the **neighborhood** $\mathcal{X}$ by

$$\mathcal{X}_\varepsilon := \{x \in \mathcal{X} \mid f(x) < f(0) + \varepsilon\} = \{0\}$$

for some $\varepsilon > 0$.

Then, ∂f is **(maximal) monotone** on $\mathcal{X}_\varepsilon \times \mathcal{V}$ with $\mathcal{V}$ is a neighborhood of 0.

Beyond convexity (II): variational convexity (con't)

Definition (f-local monotonicity of subgradient, Rockafellar VJM 19')

For lsc $f : \mathbb{R}^n \to (-\infty, +\infty]$ the mapping $\partial f : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is **f-locally monotone** around $(\bar{x}, \bar{v})$ if there is a neighborhood $\mathcal{X} \times \mathcal{V}$ of $(\bar{x}, \bar{v})$ such that

$$[\mathcal{X}_\varepsilon \times \mathcal{V}] \cap \operatorname{gph} \partial f \text{ is monotone in } \mathcal{X} \times \mathcal{V}$$

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Definition (Variational convexity, Rockafellar VJM 19')

Let $f : \mathbb{X} \to (-\infty, \infty]$ be a lsc function. f is **(strongly) variational convex** with respect to $(\bar{x}, \bar{v}) \in \operatorname{gph} \partial f$ if there exists an open convex neighborhood $\mathcal{X}$ of $\bar{x}$ and $\mathcal{V}$ of $\bar{v}$ such that there exists a proper lsc (strongly) convex function $h \leq f$ on $\mathcal{X}$ and a number $\varepsilon > 0$ such that

$$[\mathcal{X}_\varepsilon \times \mathcal{V}] \cap \operatorname{gph} \partial f = [\mathcal{X} \times \mathcal{V}] \cap \operatorname{gph} \partial h,$$

and, for any (x, v) belonging to this common set, also $h(x) = f(x)$.

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variational convexity of f $\iff$ **f-local monotonicity of ∂f**

Beyond convexity (II): variational convexity (con't)

- motivated by Poliquin **NA** 90'
- requires localization in **function values** and **subgradients**

Variational convexity vs. local convexity

- $\star$ (strong) variational convexity $\Leftarrow$ local (strong) convexity
- $\star$ (strong) variational convexity $\not\Rightarrow$ local (strong) convexity

Variational convexity vs. local convexity

⋆ (strong) variational convexity $\Leftarrow$ local (strong) convexity

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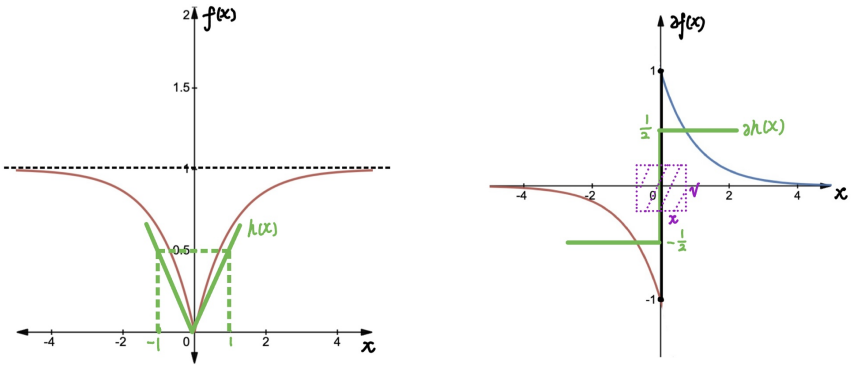
Back to **Example 1**: $f(x) = \max\{1 - e^x, 1 - e^{-x}\}$, $\partial f(x) = \begin{cases} \{-e^x\}, & x < 0 \\ [-1, 1], & x = 0 \\ \{e^{-x}\}, & x > 0 \end{cases}$



(a) (b)

- f is concave on $(-\infty, 0]$ and $[0, \infty)$
- variational convex at $\bar{x} = 0$ for $\bar{v} = 0$

  with $h(x) = \lambda|x|$ and any $\lambda \in (0, 1)$, the neighborhoods $\mathcal{X}$ and $\mathcal{V}$ being intervals adjusted for the size of λ.

(Strong) variational sufficiency

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Consider the general composite optimization problem:

$$\min_{x \in \mathbb{X}} f(x) + \theta(G(x))$$

- $f : \mathbb{X} \to \mathbb{R}$, $G : \mathbb{X} \to \mathbb{Y}$ are twice continuously differentiable
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Define

- $\phi(x, u) = f(x) + \theta(G(x) + u)$ be the **perturbed objective function** with the parameter u
- For $\rho > 0$, the **augmented objective function** $\phi_\rho(x, u) := \phi(x, u) + \frac{\rho}{2}\|u\|^2$

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Definition

The **(strong) variational sufficient condition** for local optimality holds with respect to $\bar{x}$ and $\overline{Y}$ satisfying the first order condition if there exists $\bar{\rho} > 0$ such that **augmented objective function** $\phi_{\bar{\rho}}(x, u)$ is **(strong) variational convex** with respect to the pair $\big((\bar{x}, 0), (0, \overline{Y})\big)$ in $\operatorname{gph} \partial\phi_{\bar{\rho}}$, where $\phi_{\bar{\rho}}(x, u) := \phi(x, u) + \frac{\bar{\rho}}{2}\|u\|^2$.

Local Augmented Lagrangian Duality

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Lagrangian dual

$$(D) \quad \max_Y h(Y) := \inf_x \mathcal{L}(x, Y)$$

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Local Augmented Lagrangian Duality (con't)

Theorem (Rockafellar MP 22')

Suppose **variational sufficient condition** *with respect to* $(\bar{x}, \overline{Y})$ *for local optimality is satisfied. Then, the problems* $(\widehat{P})$ *and* $(\widehat{D})$ *have optimal solutions with the same optimal value, and*

$$\bar{x} \text{ sloves } (\widehat{P}) \quad \Longleftrightarrow \quad \bar{x} \text{ minimizes in (P) relative to } \mathcal{X}.$$

Moreover, the following conditions on $(\bar{x}, \overline{Y}) \in \operatorname{int} \mathcal{X} \times \operatorname{int} \mathcal{Y}$ *are equivalent and guarantee that* $\bar{x}$ *is locally optimal relative to* $\mathcal{X}$ *in (P) with the objective value agreeing with the common optimal value in* $(\widehat{P})$ *and* $(\widehat{D})$ *as well as with* $\mathcal{L}_{\bar{\rho}}(\bar{x}, \overline{Y})$.

(a) $(\bar{x}, \overline{Y}) \in S_{KKT}(0, 0)$,

(b) $\bar{x}$ *minimize in* $(\widehat{P})$ *and* $\overline{Y}$ *maximize in* $(\widehat{D})$,

(c) $(\bar{x}, \overline{Y})$ *is a saddle point of* $\mathcal{L}_{\bar{\rho}}$ *on* $\mathcal{X} \times \mathcal{Y}$,

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can be extended to Riemannian manifold optimization: Zhou and **D.**, 2022

PPA & ALM

- **PPA for dual**

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$$Y^{k+1} \approx P_h(Y^k)$$
$$= \arg\max_Y \{h(Y) - \frac{1}{2\rho^k}\|Y - Y^k\|^2\}$$

PPA & ALM

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Nonconvex with variational sufficiency

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PPA & ALM

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$$\max_Y \{h(Y) - \frac{1}{2\rho^k}\|Y - Y^k\|^2\}$$
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Characterization of strong variational sufficiency I

Strong variational sufficiency for polyhedral

For polyhedral (Rockafellar **MP** 22'):

$$\text{strong variational sufficiency} \iff \text{strong SOSC}$$

$\star$ In particular, this case include NLP and NLSOC with $G(x) \neq 0$.

For general OP (Rockafellar **MP** 22'):

Proposition

For general optimization problem

$$\min_{x \in \mathbb{X}} f(x) + \theta(G(x)),$$

the augmented Lagrangian $\mathcal{L}_\rho$ *is* C^{1+}. *Let* $\pi_x \partial_B \big((\mathcal{L}_\rho)'_x\big)(x, Y)$ *denote the* **partial Hessian bundle**. *Then the* **strong variational sufficient condition** *for local optimality with respect to* $(\bar{x}, \overline{Y})$ *is equivalent to*

every $H \in \pi_x \partial_B \big((\mathcal{L}_\rho)'_x\big)(x, Y)$ *is* **positive definite.**

Strong variational sufficiency for general OP (con't)

Proposition (Rockafellar MP 22')

For general optimization problem

$$\min_{x \in \mathbb{X}} f(x) + \theta(G(x)),$$

strong variational sufficient condition *for local optimality with respect to* $(\bar{x}, \overline{Y})$ *is equivalent to that every* $q \in \operatorname{quad} \theta(G(\bar{x}) \mid \overline{Y})$ *has*

$$\frac{1}{2}\langle L''_{xx}(\bar{x}, \overline{Y})d, d\rangle + q(G'(\bar{x})d) > 0 \quad \text{when } d \neq 0,$$

where

$$\operatorname{quad} \theta(G(\bar{x}) \mid \overline{Y}) = \begin{cases} \text{the collection of generalized quadratic forms } q \text{ for which} \\ \exists (X^k, Y^k) \to (G(\bar{x}), \overline{Y}) \text{ with } \theta \text{ generalized twice differentiable} \\ \text{at } X^k \text{ for } Y^k \text{ and such that the generalized quadratic} \\ \text{forms } q_k = \frac{1}{2} d^2\theta(X^k \mid Y^k) \text{ converge epigraphically to } q. \end{cases}$$

is the **quadratic bundle** *of* θ.

Our result: toy example

Consider the following NLSOC:

$$\min f(x) + \delta_{\mathcal{K}}(g(x)),$$

where $\mathcal{K} = \{(u_1, \ldots, u_m) \in \mathbb{R}^m \mid u_1 \geq \|(u_2, \ldots, u_m)\|\} = \{u \mid h(u) \leq 0\}$.

$$\mathcal{M}_{\text{soc}}(\bar{x}) := \left\{ y \in \mathbb{R}^m \,\middle|\, \begin{array}{l} L'_x(\bar{x}, y) = 0, \\ \mathcal{K} \ni g(\bar{x}) \perp y \in \mathcal{K}^\circ \end{array} \right\}.$$

Proposition

Let $\bar{x} \in \mathbb{X}$ be a stationary point to the NLSOC and $\bar{y} \in \mathcal{M}_{SOC}(\bar{x})$. The **strong variational sufficient condition** *with respect to $(\bar{x}, \bar{y})$ holds if and only if the* **strong second order sufficient condition** *holds at $(\bar{x}, \bar{y})$, i.e.,*

$$\langle L''_{xx}(\bar{x}, \bar{y})d, d\rangle - \Upsilon_{g(\bar{x})}\big(\bar{y}, g'(\bar{x})d\big) > 0 \quad \forall\, 0 \neq g'(\bar{x})d \in \operatorname{aff} \mathcal{C}_{\mathcal{K}}(g(\bar{x}), \bar{y}),$$

where $\Upsilon_{g(\bar{x})}\big(\bar{y}, g'(\bar{x})d\big)$ is the so-called "σ-term".

Our result: toy example (con't)

Main idea: calculating the quadratic bundle for NLSOC:

- when $g(\bar{x}) \neq 0 \Longrightarrow$ polyhedral case
- when $g(\bar{x}) = 0 \Longrightarrow$ the $\sigma-$term happens to be 0, which makes the calculation much simpler

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$\star$ **main tool** (Mohammadi, Mordukhovich and Sarabi **TAMS** 20'): Suppose $v \in \mathcal{N}_{\mathcal{K}}(x)$,

$$d^2\delta_{\mathcal{K}}(x \mid v)(w) = -\Upsilon_x\big(v, w\big) + \delta_{\mathcal{C}_{\mathcal{K}}(x,v)}(w)$$

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Extension of Rockafellar **MP** 22' in which only consider the case $g(\bar{x}) \neq 0$

However,

what about **NLSDP? general MOP?**

Our results: for NLSDP

Consider the following NLSDP:

$$\min f(x) + \delta_{\mathbb{S}^n_+}(G(x))$$

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*where $\Upsilon_{G(\bar{x})}\big(\overline{Y}, G'(\bar{x})d\big) := 2\langle \overline{Y}, (G'(\bar{x})d)G(\bar{x})^\dagger(G'(\bar{x})d)\rangle$ is the so-called "σ-**term**" and $G(\bar{x})^\dagger$ is the generalized inverse matrix of $G(\bar{x})$.*

Comparison with Khanh, Mordukhovich and Phat's approach

Recently, Khanh, Mordukhovich and Phat 22' provides a sufficient condition for strong variational sufficiency under the following second order qualification:

$$\ker G'(\bar{x})^* \cap \partial^2\theta(G(\bar{x}), \overline{Y})(0) = \{0\},$$

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Proposition

Let $\bar{x} \in \mathbb{X}$ be a stationary point to NLSDP and $\overline{Y}$ be a corresponding multiplier. Then, the second order qualification condition at $(G(\bar{x}), \overline{Y})$ is equivalent to the nondegeneracy condition (LICQ) at $(G(\bar{x}), \overline{Y})$, i.e.,

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Proof sketch: We only need to verify $\operatorname{lin}\big(\mathcal{T}_{\mathbb{S}^n_+}(G(\bar{x}))\big)^\perp = \partial^2\theta(G(\bar{x}), \overline{Y})(0)$, which can be obtained directly from Ding, Sun and Ye **MP** 14'.

Comparison with Khanh, Mordukhovich and Phat's approach (con't)

Khanh, Mordukhovich and Phat 22':

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$$\text{strong SOSC} + \text{LICQ} \quad \Longrightarrow \quad \text{strong variational sufficiency}$$

This work: without any constraint qualification or the uniqueness of multiplier

$$\text{strong SOSC} \quad \Longleftrightarrow \quad \text{strong variational sufficiency}$$

ALM for NLSDP

Inexact ALM for non-convex NLSDP

Algorithm 1 Inexact ALM for non-convex NLSDP

Input: Let $(x^0, Y^0) \in \mathbb{X} \times \mathbb{S}^n$, $\rho^0, \rho_0^0 > 0$. Set $k := 0$. $\{\epsilon_k\}_{k \geq 0}$ with $\epsilon_k > 0$ for all k and $\epsilon_k \to 0$ and set $k := 0$.

Output: x^{k+1} and Y^{k+1}

1: If (x^k, Y^k) satisfies a suitable termination criterion: STOP.

2: Compute x^{k+1} such that

$$x^{k+1} \approx \bar{x}^{k+1} = \operatorname{argmin}_{x \in \mathcal{X}} \mathcal{L}_{\rho^k}(x, Y^k),$$

3: Update the vector of multipliers to

$$Y^{k+1} := Y^k + \rho_0^k (\mathcal{L}_{\rho^k})'_Y (x^{k+1}, Y^k),$$

where $\rho_0^k = \rho^k - \bar{\rho}$ and $\bar{\rho} > 0$.

4: Update nondecreasing positive sequence ρ^{k+1} according to certain rules.

5: Set $k \leftarrow k + 1$ and go to **Step 1**.

Inexact ALM for non-convex NLSDP

Algorithm 2 Inexact ALM for non-convex NLSDP

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Under strong variational sufficiency, an **implementable** stopping criterion for the updating of x^{k+1}:

$$\sqrt{\rho_0^k}\|(\mathcal{L}_{\rho^k})'_x(x^{k+1}, Y^k)\| \leq \begin{cases} (a)\ \epsilon'_k, \\ (b)\ \epsilon'_k \min\{1, \|\rho_0^k(\mathcal{L}_{\rho^k})'_Y(x^{k+1}, Y^k)\|\}, \\ (c)\ \epsilon'_k \min\{1, \|\rho_0^k(\mathcal{L}_{\rho^k})'_Y(x^{k+1}, Y^k)\|^2\}, \end{cases}$$

where $\epsilon'_k = \epsilon_k\sqrt{s}$ for some $s > 0$.

Bounded linear regularity

Definition

Let $D_1, D_2, \ldots, D_m \subseteq \mathbb{X}$ be closed convex sets for some positive integer m. Suppose that $D := D_1 \cap D_2 \cap \ldots \cap D_m$ is non-empty. The collection $\{D_1, D_2, \ldots, D_m\}$ is said to be boundedly linearly regular if for every bounded set $\mathcal{B} \subseteq \mathbb{X}$, there exists a constant $\kappa > 0$ such that

$$\operatorname{dist}(x, D) \leqslant \kappa \max\left\{\operatorname{dist}\left(x, D_1\right), \ldots, \operatorname{dist}\left(x, D_m\right)\right\} \quad \forall x \in \mathcal{B}.$$

$$\mathcal{G}_1(\bar{x}) = \{Y \in \mathbb{S}^n \mid f'(\bar{x}) + G'(\bar{x})^* Y = 0\}$$

$$\mathcal{G}_2(\bar{x}) = \{Y \in \mathbb{S}^n \mid Y \in \mathcal{N}_{\mathbb{S}^n_+}(G(\bar{x}))\}$$

New local convergence results of ALM for NLSDP

Theorem (Local convergence of ALM for NLSDP)

Let $\bar{x} \in \mathcal{X}$ be a stationary point to the NLSDP with a multiplier $\overline{Y} \in \mathcal{M}(\bar{x})$. Suppose the **strongly SOSC** *holds at $(\bar{x}, \overline{Y})$. Let Y^0 be an initial point sufficiently close to $\mathcal{M}(\bar{x})$. Suppose the set $\{x \mid -(\mathcal{L}_{\bar{\rho}})'_x(x, Y) \in \mathcal{N}_{\mathcal{X}}(x)\}$ is nonempty and bounded when $Y \in \operatorname{int} \mathcal{Y}$.*

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(i) *Under stopping criterion (a): the sequence $\{Y^k\}$ converge to a particular solution $\widehat{Y} \in \mathcal{M}(\bar{x})$ and $x^k \to \bar{x}$.*

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(i) *Under stopping criterion (a): the sequence $\{Y^k\}$ converge to a particular solution $\widehat{Y} \in \mathcal{M}(\bar{x})$ and $x^k \to \bar{x}$.*

(ii) *Under stopping criterion (b): suppose $\{\mathcal{G}_1(\bar{x}), \mathcal{G}_2(\bar{x})\}$ is also* **boundedly linearly regular**. *Then, we have $\operatorname{dist}(Y^k, \mathcal{M}(\bar{x})) \to 0$ with*

$$\operatorname{dist}(Y^{k+1}, \mathcal{M}(\bar{x})) \leq \frac{1}{\sqrt{1 + b^2(\rho^\infty)^2}} \operatorname{dist}(Y^k, \mathcal{M}(\bar{x}))$$

and $\bar{x}^k \to \bar{x}$ with

$$\|\bar{x}^k - \bar{x}\| \leq \frac{1}{s} \operatorname{dist}(Y^k, \mathcal{M}(\bar{x})),$$

where $\bar{x}^k$ **is the exact solution of ALM subproblems**.

New local convergence results of ALM for NLSDP (con't)

(iii) Under stopping criterion (c): the sequence $\{Y^k\}$ converge to a particular solution $\widehat{Y} \in \mathcal{M}(\bar{x})$ with

$$\|Y^{k+1} - \widehat{Y}\| \leq \frac{1}{\sqrt{1 + b^2(\rho^\infty)^2}} \|Y^k - \widehat{Y}\|.$$

Moreover, if the stopping criterion is further supplemented by

$$\|(\mathcal{L}_{\rho^k})'_x(x^{k+1}, Y^k)\| \leq c\|Y^{k+1} - Y^k\| \text{ for some fixed } c,$$

we have $x^k \to \bar{x}$ with

$$\|x^k - \bar{x}\| \leq p\|Y^k - \widehat{Y}\|$$

for some $p > 0$.

New local convergence results of ALM for NLSDP (con't)

Comments:

- The bounded linear regularity of the collection $\{\mathcal{G}_1(\bar{x}), \mathcal{G}_2(\bar{x})\}$ holds under one of the following two conditions:

  (i) $\mathcal{G}_2(\bar{x})$ is a polyhedron, i.e., $|\gamma| \geq n-1$, where γ is the negative eigenvalue index set of matrix $G(\bar{x}) + \overline{Y}$;

  (ii) there exists a strict complementarity KKT pair $(\bar{x}, \widetilde{Y})$ ($\widetilde{Y}$ does not have to be $\overline{Y}$), i.e., $\mathrm{rank}(G(\bar{x})) + \mathrm{rank}(\widetilde{Y}) = n$.

- $\bar{x}^k$ is usually unknown.

R-linear convergence of the KKT residual

Inspired by Cui, Sun and Toh **MP** 19', we obtain the following R-linear convergence of the KKT residual.

R-linear convergence of the KKT residual

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Theorem

Define the following residual function

$$R(x, Y) := \|L'_x(x, Y)\| + \|G(x) - \Pi_{\mathbb{S}^n_+}(G(x) + Y)\|.$$

Then, under stopping criterion (b), for k sufficiently large, if $\sqrt{\rho_0^k}\epsilon_k < 1$, we have there exists $a > 0$ such that

$$R(x^{k+1}, Y^{k+1}) \leq c^k \mathrm{dist}(Y^k, \mathcal{M}(\bar{x}))$$

with $c^k = (\epsilon'_k\sqrt{\rho_0^k} + (1 + \bar{\rho} + a\bar{\rho})(\rho_0^k)^{-1})(1 - \sqrt{\rho_0^k}\epsilon_k)^{-1}$.

Summary

- For **non-convex** and **non-polyhedral** problems (**NLSDP**, **NLSOC**)

	optimality	CQs
Sun, Sun and Zhang **MP** 08'	Strong SOSC	LICQ
Kanzow and Steck **MP** 19'	SOSC	SRCQ
Wang and **D.** 21'	SOSC	partially free+
This work	Strong SOSC	free+

Characterization of strong variational sufficiency II

Solving ALM subproblem: semi-smooth Newton CG method

ALM subproblem:

$$\min_{x \in \mathcal{X}} \ \mathcal{L}_\rho(x, Y)$$

with

$$\mathcal{L}_\rho(x, Y) := f(x) + \frac{\rho}{2} \|\Pi_{\mathbb{S}_-}(G(x) + \frac{Y}{\rho})\|^2 - \frac{\|Y\|^2}{2\rho}$$

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B(ouligand)-subdifferential of $(\mathcal{L}_\rho)'_x$:

$$\partial_B\big((\mathcal{L}_\rho)'_x\big)(x, Y) := \{\lim_{k \to \infty} (\mathcal{L}_\rho)''_{xx}(x^k, Y^k) \mid (x^k, Y^k) \in \mathcal{U},\ (x^k, Y^k) \to (x, Y)\},$$

where $\mathcal{U}$ is the set of F-differentiable points of $(\mathcal{L}_\rho)'_x$

Semi-smooth Newton CG method

- Sun, Sun, Zhang **MP** 08':

$$\pi_x \partial_B \big((\mathcal{L}_\rho)'_x \big)(x, Y)(\Delta x)$$
$$= L''_{xx}(x, \Pi_{\mathbb{S}^n_-}(G(x) + \rho^{-1} Y))(\Delta x) + \rho G'(x)^* \partial_B \Pi_{\mathbb{S}^n_-}(G(x) + \rho^{-1} Y) G'(x)(\Delta x)$$

with

  - $\pi_x(\cdot)$: the projection onto the space $\mathbb{X}$
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- For any $W \in \partial_B \Pi_{\mathbb{S}^n_-}(G(x) + \rho^{-1}Y)$, define the following matrix

$$\mathcal{A}_\rho(Y, W) := L''_{xx}(x, \Pi_{\mathbb{S}^n_-}(G(x) + \rho^{-1}Y)) + \rho G'(x)^* W G'(x) \in \pi_x \partial_B \big((\mathcal{L}_\rho)'_x\big)(x, Y)$$

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Newton equation:

$$\mathcal{A}_\rho(Y,W)d = -(\mathcal{L}_\rho)'_x(x,Y)$$

The non-singularity of $\mathcal{A}_\rho(Y, W)$

Rockefeller **MP** 22':

- $\pi_x \partial_B \big((\mathcal{L}_\rho)'_x\big)(x, Y)$: the (partial) **Hessian bundle** for $\mathcal{L}_\rho$

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- **Hessian bundle criterion**:

$$\textbf{Strong variational sufficiency} \quad \Longleftrightarrow \quad \pi_x \partial_B \big((\mathcal{L}_\rho)'_x\big)(x, Y) \succ 0$$

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Theorem (Characterization of strong variational sufficiency for NLSDP)

Let $\bar{x} \in \mathcal{X}$ be a stationary point to the NLSDP and $\overline{Y}$ be a corresponding multiple. Then the following conditions are equivalent.

(i) *The* **strong variational sufficient condition** *with respect to $(\bar{x}, \overline{Y})$ holds.*

(ii) *The* **strong second order sufficient condition** *holds at $(\bar{x}, \overline{Y})$.*

(iii) *there exist $\rho_0 > 0$ and $\underline{\eta} > 0$ such that for any $\rho \geq \rho_0$ and any $W \in \partial_B \Pi_{\mathbb{S}^n_-}(G(\bar{x}) + \rho^{-1}\overline{Y})$,*

$$\langle d, \mathcal{A}_\rho(\overline{Y}, W) d \rangle \geq \underline{\eta} \|d\|^2 \quad \forall d \in \mathbb{X},$$

The non-singularity of $\mathcal{A}_\rho(Y, W)$ (con't)

Proof sketch:

(i)$\Longrightarrow$(ii): Done.

Proposition

Let $\bar{x} \in \mathbb{X}$ be a stationary point to NLSDP and $\overline{Y} \in \mathcal{M}(\bar{x})$. Then, there exists $q \in \operatorname{quad} \delta_{\mathbb{S}^n_+}(G(\bar{x}) \mid \overline{Y})$ such that for all $H \in \mathbb{S}^n$,

$$\begin{aligned} q(H) &= -\frac{1}{2}\Upsilon_{G(\bar{x})}\big(\overline{Y}, H\big) + \delta_{\operatorname{aff} \mathcal{C}_{\mathbb{S}^n_+}(G(\bar{x}),\overline{Y})}(H) \\ &= \sum_{i\in\alpha, j\in\gamma} \frac{-\lambda_j(A)}{\lambda_i(A)}(\widetilde{H}_{ij})^2 + \delta_{\operatorname{aff} \mathcal{C}_{\mathbb{S}^n_+}(G(\bar{x}),\overline{Y})}(H), \end{aligned}$$

where $\widetilde{H} = P^T H P$.

(ii)$\Longrightarrow$(iii): Similar to Sun, Sun and Zhang **MP** 08', Proposition 4

by applying variational properties of SDP cone

(iii)$\Longrightarrow$(i): Directly by Rockafellar **MP** 22'

Augmented tilt stability

Definition

In context of the local (strong) convexity-concavity of $\mathcal{L}_\rho(x, Y)$, the **augmented tilt stability** will be said to hold if there is a neighborhood $\mathcal{V}$ of 0 such that the mapping

$$(v, Y) \mapsto \arg\min_{x \in \mathcal{X}} \{\mathcal{L}_\rho(x, Y) - \langle v, x \rangle\} \quad \text{for} \quad (v, y) \in \mathcal{V} \times \mathcal{Y}$$

is single-valued and Lipschitz continuous.

Sun, Sun and Zhang **MP** 08':

SSOSC+LICQ $\Rightarrow$ positive definiteness of $\mathcal{A}_\rho(\overline{Y}, W)$ + Assumption B2 (directional derivative estimation of $x(Y)$)

$\Rightarrow$ **augmented tilt stability** with $v = 0$ for NLSDP

Rockafellar **MP** 22':

strong variational sufficiency $\Rightarrow$ **augmented tilt stability** for general OP

Numerical experiments

A toy example

Consider the following **non-convex** NLSDP:

$$\begin{array}{ll} \min & \frac{1}{2}x^3 \\ \text{s.t} & -x^2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathcal{S}^3_+ \quad \Leftarrow \quad \Gamma \end{array}$$

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- $\bar{x} = 0$ is the optimal solution
- The corresponding multiplier set $\mathcal{M}(\bar{x}) := \{\Gamma \mid \Gamma \in \mathcal{S}^3_-\}$
- Robinson's CQ (MFCQ) does not hold
- Let $\overline{\Gamma} = \mathrm{Diag}(0, -1, -2)$. It is easy to verify $L''_{xx}(\bar{x}, \overline{\Gamma}) = 4 > 0$, which implies strongly second order sufficient condition holds at $(\bar{x}, \overline{\Gamma})$.

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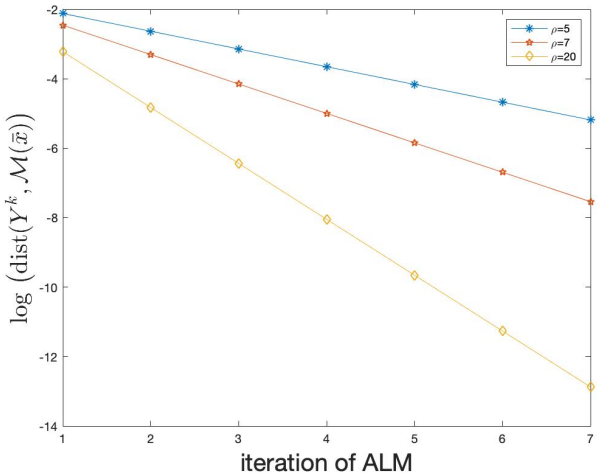


A slightly complicated example

$$\begin{array}{ll} \min\limits_{X \in \mathbb{S}^n} & \frac{1}{2}\langle X, Q \circ X \rangle \\ \text{s.t.} & B \circ X = 0 \\ & X \in \mathbb{S}^n_+, \end{array}$$

where "$\circ$" denotes the Hadamard product of matrices,

$$B = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 \\ 1 & \cdots & 1 & 1 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} q & \cdots & 1 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 1 & \cdots & q & 0 \\ 0 & \cdots & 0 & -1 \end{bmatrix}$$

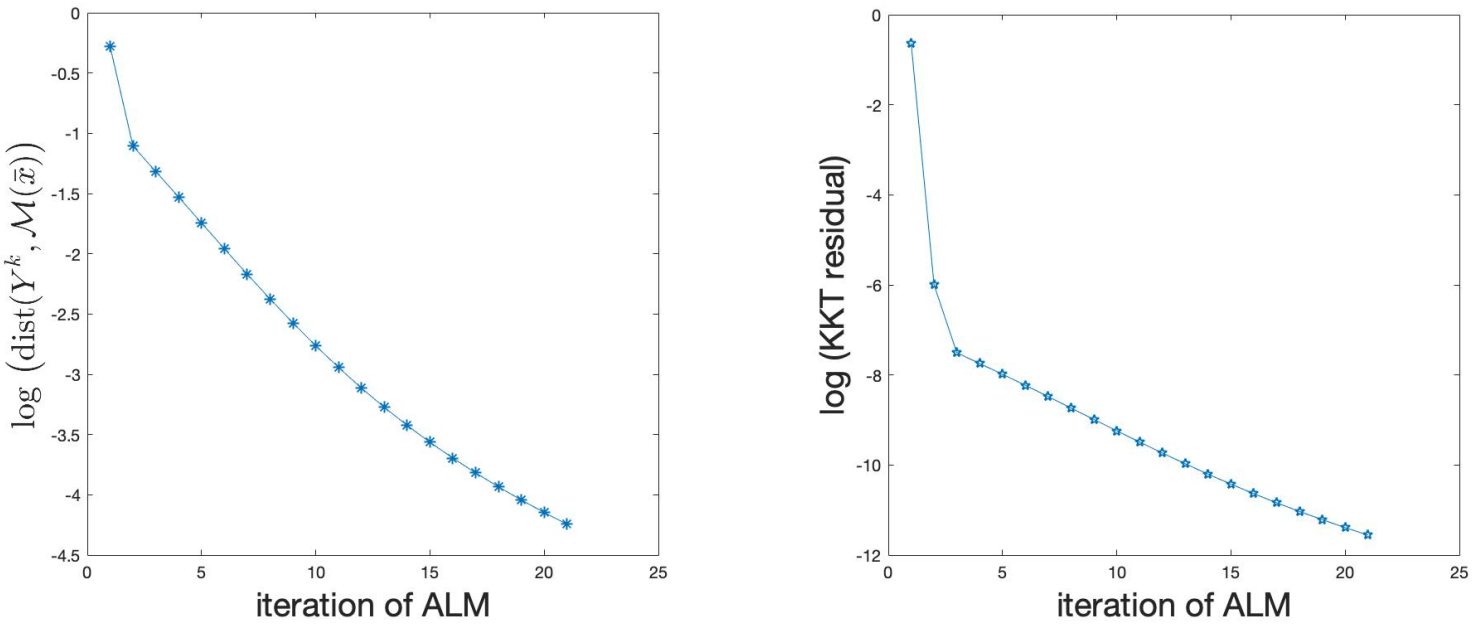
with a given $q > n - 1$.

- $\overline{X} = 0$ is a local optimal solution
- the corresponding multiplier set is given by

$$\mathcal{M}(\overline{X}) = \{(Y, Z) \in \mathbb{S}^n \times \mathbb{S}^n \mid Y + B \circ Z = 0, Y \in \mathbb{S}^n_-\}.$$

- Robinson's CQ does not hold
- strongly SOSC holds and $\exists\, \widehat{Y} \in \mathcal{M}(\bar{x})$ s.t. $(\bar{x}, \widehat{Y})$ is a strict complementarity KKT pair

A slightly complicated example (con't)



(a) (b)

Figure 2: Semismooth Newton-CG based ALM for problem with $n = 1000$ and $q = 1500$

The non-singularity of Hessian V: e.g.,

- $\lambda_{\min}(V) \approx 1.01$ if $n = 3$ and $q = 2$
- $\lambda_{\min}(V) \approx 1.26$ if $n = 100$ and $q = 200$

A slightly complicated example (con't)

n	q	iteration	KKT residual	cpu(s)
3	2	8	8.27e-06	0.22
100	200	11	8.98e-06	3.11
1000	1500	21	9.57e-06	1083.34

Table 1: Numerical results of semismooth Newton-CG based ALM

Conclusions

In this talk:

- The explicit characterization of strong variational sufficiency without any CQ
- Convergence rate of ALM for **NLSDP**
- Simple numerical experiments are conducted to verify our theory

Similar results can be obtained for the general matrix optimization.

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ALM for non-smooth optimization on Riemannian manifolds :

- Y.H. Zhou, C.L. Bao, **D.**, and J. Zhu, A semismooth Newton based augmented Lagrangian method for nonsmooth optimization on matrix manifolds, **MP**, 1-61 (2022).
- Y.X. Zhou, C.L. Bao and **D.**, On the robust isolated calmness of a class of nonsmooth optimizations on Riemannian manifolds and its applications, 2022, arXiv: 2208.07518.

Thank you!